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TATM38, Exercises on time discrete models

1. [$\sim$ 1.6a in EK] Solve $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ by using eigenvalues and eigenvectors.
Solve also the system by rewriting it as an order-2 difference equation for x_n.

2. Solve $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ by using eigenvalues and eigenvectors.
What is the solution if we have initial conditions $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?
Write down $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ for $n = 0, 1, 2, \ldots, 8$ for these initial values.

3. [2.3 in EK] For the Ricker model for a fish population N_n at time n,

$$N_{n+1} = \alpha N_n e^{-\beta N_n} ,$$

$\alpha > 0$ and $\beta > 0$ constants, find the steady states and determine their stability (depending on α and β).

4. [2.4 in EK] Consider the population model (see EK for interpretation)

$$N_{n+1} = N_n e^{r(1-N_n/K)} ,$$

with $r > 0$ and $K > 0$ constants.
Plot the function $g(N) = e^{r(1-N/K)}$. For what values of N_n is $N_{n+1} > N_n$?
How can we interpret K?
Find the steady states and determine their stability (depending on α and β).
Let $K = 1$. Take some initial values N_0, e.g., $N_0 = 2$ and $N_0 = 0.9$, and calculate some values N_1, N_2, N_3, N_4, ... for $r = 1$ and $r = 3$.

5. A time discrete predator-prey model is given by the system

$$\begin{cases} x_{n+1} = x_n + r x_n(1 - x_n) - 2 x_n y_n \\ y_{n+1} = 0.5 y_n + x_n y_n \end{cases}$$

Here x_n and y_n are the numbers of prey and predators, respectively, at time n and y_n, and $0 < r < 4$ is a constant. Find all steady states and determine their stability.
Extra: make a plot of x_n and y_n for some value of r and some initial values

6. The time discrete epidemic SIRS model is given by the system

$$\begin{cases} S_{n+1} = S_n - \beta S_n I_n + \gamma R_n \\ I_{n+1} = I_n - \nu I_n + \beta S_n I_n \\ R_{n+1} = R_n - \gamma R_n + \nu I_n \end{cases}$$

Here S_n, I_n and R_n are the numbers of susceptible, infected and removed, respectively, at time n.
$\beta > 0$, $\nu > 0$ and $\gamma > 0$ are constants.
Use that $S_n + I_n + R_n = N$ (= constant = the whole poulation) to write a 2D system for S_n and I_n.
For simplicity, assume now that $\nu = \gamma = 0.5$ and $N = 100$.
Find all steady states and determine their stability.
For $\beta = \frac{1}{250}$ ($\Rightarrow R_0 = \frac{N\beta}{\nu} = 0.8$), what are the steady states and their stability?
For $\beta = \frac{1}{100}$ ($\Rightarrow R_0 = 2$), what are the steady states and their stability?
Make some plots of S_n and I_n for these values of β and some initial values.

7. [$\sim$ 2.2b in EK] Consider the model

$$x_{n+1} = -x_n^2(1 - x_n)$$

Find all steady states and determine their stability.
Sketch a cobweb diagram for some x_0. What happens to x_n as $n \to \infty$ for different x_0?

8. [$\sim$ 2.8bc in EK] Consider the population model

$$N_{n+1} = \frac{\lambda N_n}{(1 + N_n)^2} \, ,$$

where $\lambda > 0$ is constant, and $N_n \geq 0$ always.
Find all steady states and determine their stability.
Sketch a cobweb diagram for $\lambda = 2$. What happens to N_n as $n \to \infty$ for different $N_0 \geq 0$?

9. A population is divided into three age classes. The number of individuals at time n in the classes are x_n, y_n and z_n, where x_n is the youngest class and z_n the oldest. The survival rate from x_n to y_n is 1/2, and from y_n to z_n 1/3. The average number of births from individuals in the three classes are 0, 3/2 and 3/2, respectively. Write down a linear system with a Leslie matrix for the time evolution of the populations of the different age classes. What is the approximate age distribution of the population for large times n (i.e., what fractions of the entire population are in the different age classes x_n, y_n and z_n) ?

Solutions (1-2 typed, 3-9 handwritten)

1. Eigenvalues of $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 2$, and corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Solutions are $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1\, 5^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\, 2^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Finding the order-2 difference equation for x_n:
$x_{n+1} = 3x_n + 2y_n$, $y_{n+1} = x_n + 4y_n \;\Rightarrow\; x_{n+2} = 3x_{n+1} + 2y_{n+1} = 3x_{n+1} + 2(x_n + 4y_n) =$
$= 3x_{n+1} + 2x_n + 4(x_{n+1} - 3x_n) = 7x_{n+1} - 10x_n \;\Rightarrow$
$x_{n+2} - 7x_{n+1} + 10x_n = 0$, which has solution $x_n = k_1 5^n + k_2 2^n$.
Then $y_n = \frac{1}{2}(x_{n+1} - 3x_n) = \frac{1}{2}(5k_1 5^n + 2k_2 2^n - 3k_1 5^n - 3k_2 2^n) = k_1 5^n - \frac{1}{2}k_2 2^n$,
the same as above ($k_1 = c_1$, $k_2 = 2c_2$).

2. Eigenvalues of $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ are $\lambda_{1,2} = 1 \pm i = \sqrt{2}e^{\pm i\pi/4}$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$.

Solutions are $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1\,(1+i)^n \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2\,(1-i)^n \begin{pmatrix} 1 \\ -i \end{pmatrix} = 2^{n/2} \begin{pmatrix} k_1 \cos\frac{n\pi}{4} + k_2 \sin\frac{n\pi}{4} \\ -k_1 \sin\frac{n\pi}{4} + k_2 \cos\frac{n\pi}{4} \end{pmatrix}$,
where $k_1 = c_1 + c_2$ and $k_2 = i(c_1 - c_2)$.

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow k_1 = 1,\, k_2 = 0 \Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^{n/2} \begin{pmatrix} \cos\frac{n\pi}{4} \\ -\sin\frac{n\pi}{4} \end{pmatrix}$

We get

$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x_5 \\ y_5 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$,
$\begin{pmatrix} x_6 \\ y_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$, $\begin{pmatrix} x_7 \\ y_7 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$, $\begin{pmatrix} x_8 \\ y_8 \end{pmatrix} = \begin{pmatrix} 16 \\ 0 \end{pmatrix}$,

(3) $N_{n+1} = \alpha N_n e^{-\beta N_n} = f(N_n)$, $\alpha>0, \beta>0$

Steady states: $\bar{N} = \alpha \bar{N} e^{-\beta \bar{N}} \Rightarrow \bar{N}_1 = 0$ or $e^{\beta \bar{N}_2} = \alpha \Rightarrow \bar{N}_2 = \frac{\ln\alpha}{\beta}$

$\bar{N}_2 > 0$ requires $\alpha > 1$ to exist.

Stability: $f'(N) = \alpha(1-\beta N)e^{-\beta N} \Rightarrow$

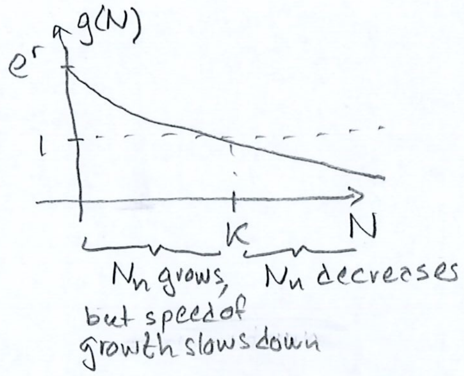
$|f'(0)| = \alpha \Rightarrow \begin{cases} \text{stable if } \alpha<1\text{, then there is no } \bar{N}_2 \text{ and } N_n \to 0, n\to\infty \\ \quad \text{(population dies out)} \\ \text{unstable if } \alpha>1 \end{cases}$

$|f'(\frac{\ln\alpha}{\beta})| = |(\underbrace{1-\beta\bar{N}_2}_{=\ln\alpha}) \underbrace{\alpha e^{-\beta\bar{N}_2}}_{=1}| = |1-\ln\alpha| < 1$ if $0 < \ln\alpha < 2 \Leftrightarrow$

$1 < \alpha < e^2 \Rightarrow \bar{N}_2$ is stable if $1<\alpha<e^2$.

If $\alpha > e^2$ there is no stable steady state

(4) $N_{n+1} = N_n \underbrace{e^{r(1-N_n/K)}}_{g(N_n)}$



K is a carrying capacity.

Increases if $N_n < K$
Decreases if $N_n > K$

Steady states:

$\bar{N} = \bar{N} e^{r(1-\bar{N}/K)} \Rightarrow \bar{N}_1 = 0$ or $r(1-\bar{N}_2/K) = 0 \Rightarrow \bar{N}_2 = K$

Stability: $f(N) = N e^{r-rN/K} \Rightarrow f'(N) = (1-\frac{rN}{K})e^{r-rN/K} \Rightarrow$

$|f'(0)| = e^r > 1$ (since $r>0$) $\Rightarrow \bar{N}_1 = 0$ unstable

$|f'(K)| = |(1-r)\cdot e^0| = |1-r|$; stable if $|1-r|<1 \Leftrightarrow \underline{0<r<2}$

For $0<r<2$, $N_n \to K$ if $N_0 > 0$

Tests; $K=1$ ($\Rightarrow \bar{N}_2 = 1$)

	$r=1$ (stable)		$r=3$ (unstable)	
N_0	2	0.9	2	0.9
N_1	0.74	0.995	0.10	1.21
N_2	0.96	0.99999	1.48	0.64
⋮	0.99	⋮	0.35	1.89
	↓	↓	2.46	0.13
	1	1	0.03	1.77
			0.56	⋮
			⋮	⋮

no limits, predictable?

5 $\begin{cases} x_{n+1} = x_n + r x_n(1-x_n) - 2x_n y_n \\ y_{n+1} = 0.5 y_n + x_n y_n \end{cases}$ $\quad 0<r<4$

Steady states

$$\begin{cases} \bar{x} = \bar{x} + r\bar{x}(1-\bar{x}) - 2\bar{x}\bar{y} \\ \bar{y} = 0.5\bar{y} + \bar{x}\bar{y} \end{cases} \iff$$

$$\begin{cases} \bar{x}\left(r(1-\bar{x}) - 2\bar{y}\right) = 0 \quad (1) \\ \bar{y}(0.5-\bar{x}) = 0 \quad (2) \Rightarrow \bar{y}=0 \text{ or } \bar{x}=0.5 \end{cases}$$

$\bar{y}=0$ in (1) $\Rightarrow \bar{x}=0$ or $\bar{x}=1$

$\bar{x}=0.5$ in (1) $\Rightarrow \bar{y}=\frac{r}{4}$

3 steady states $(\bar{x}_1,\bar{y}_1)=(0,0)$, $(\bar{x}_2,\bar{y}_2)=(1,0)$, $(\bar{x}_3,\bar{y}_3)=(0.5,\frac{r}{4})$

Stability

$$J(x,y) = \begin{pmatrix} 1+r-2rx-2y & -2x \\ y & 0.5+x \end{pmatrix} \Rightarrow$$

$$J(0,0) = \begin{pmatrix} 1+r & 0 \\ 0 & 0.5 \end{pmatrix} \quad \lambda_1 = 1+r > 1 \Rightarrow \underline{\text{unstable}}$$

$$J(1,0) = \begin{pmatrix} 1-r & -2 \\ 0 & 1.5 \end{pmatrix} \quad \lambda_1 = 1-r, \ \lambda_2 = 1.5 > 1$$

$\Downarrow$

$\underline{\text{unstable}}$

$$J(0.5,\tfrac{r}{4}) = \begin{pmatrix} 1-\frac{r}{2} & -1 \\ \frac{r}{4} & 1 \end{pmatrix} \Rightarrow \lambda^2 - (2-\tfrac{r}{2}\lambda) + 1 - \tfrac{r}{4} = 0 \Rightarrow$$

$$\lambda_{1,2} = 1-\frac{r}{4} \pm \sqrt{1-\frac{r}{2}+\frac{r^2}{16}-1+\frac{r}{4}} = 1-\frac{r}{4} \pm \sqrt{\frac{r^2}{16}-\frac{r}{4}} =$$

$$= 1-\frac{r}{4} \pm \frac{1}{4}\sqrt{\underbrace{r(r-4)}_{<0}} = 1-\frac{r}{4} \pm \frac{i}{4}\sqrt{r(4-r)}$$

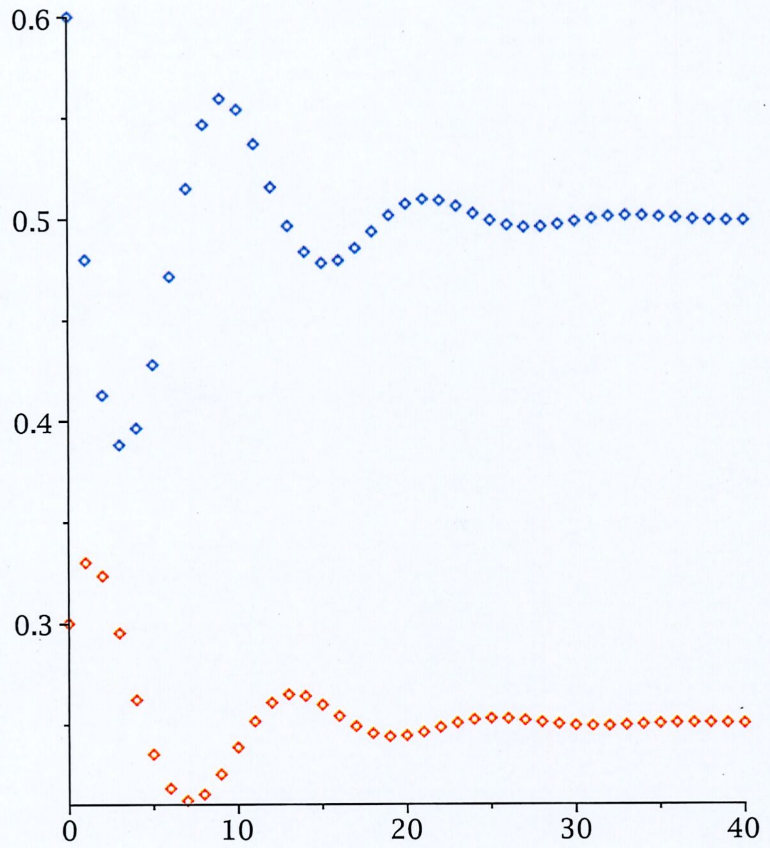
$$\Rightarrow |\lambda_{1,2}|^2 = \left(1-\frac{r}{4}\right)^2 + \frac{1}{16}r(4-r) = 1-\frac{r}{4} < 1 \Rightarrow |\lambda_{1,2}| < 1 \Rightarrow \underline{\text{stable}}$$

See Maple plots for * $r=1$, $(x_0,y_0)=(0.6,0.3)$, $(x_n,y_n) \xrightarrow[n\to\infty]{} (0.5,0.25) = (\bar{x}_3,\bar{y}_3)$,

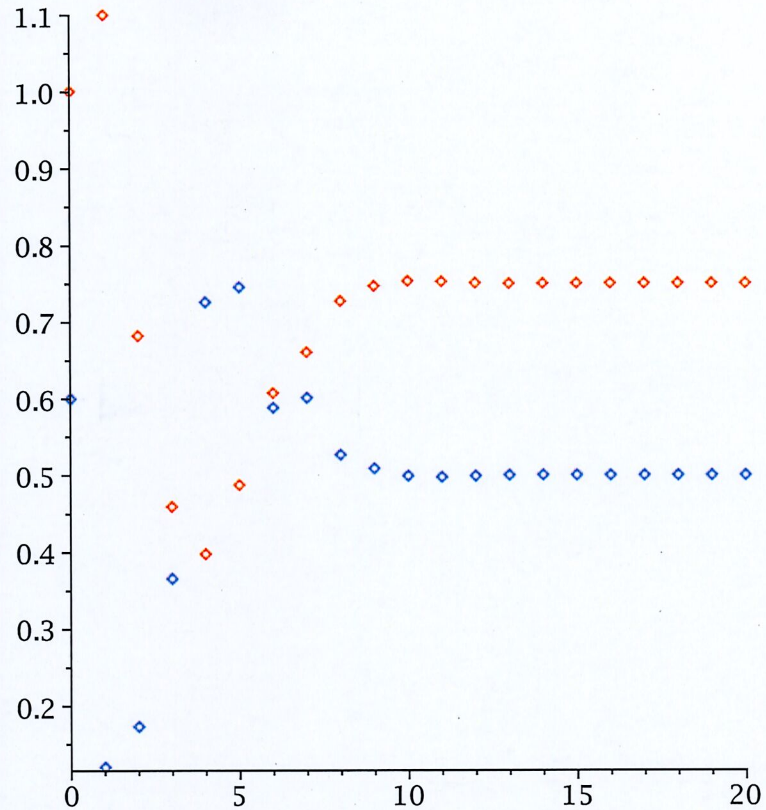
* $r=3$, $(x_0,y_0)=(0.6,1)$; $(x_n,y_n) \xrightarrow[n\to\infty]{} (0.5,0.75) = (\bar{x}_3,\bar{y}_3)$,

Note that if (x_0,y_0) too far from $(\bar{x}_3,\bar{y}_3)$, (x_n,y_n) will not converge to $(\bar{x}_3,\bar{y}_3)$.

```
> x[0] := 0.6 : y[0] := 0.3 : r := 1 :
> for n from 0 to 39 do
    x[n+1] := x[n]·(1 + r − r·x[n] − 2· y[n]);
    y[n+1] := y[n]·(0.5 + x[n]);
  od:
> with(plots) :
> A := pointplot([seq([n, x[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, y[n]], n = 0..40)], color = red) :
> display(A, B)
```




```
> x[0] := 0.6 : y[0] := 1 : r := 3 :
> for n from 0 to 19 do
    x[n+1] := x[n]·(1 + r − r·x[n] − 2· y[n]);
    y[n+1] := y[n]·(0.5 + x[n]);
  od:
> A := pointplot([seq([n, x[n]], n = 0..20)], color = blue) :
> B := pointplot([seq([n, y[n]], n = 0..20)], color = red) :
> display(A, B)
```

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⑥ $\begin{cases} S_{n+1} = S_n - \beta S_n I_n + \gamma R_n \\ I_{n+1} = I_n - \nu I_n + \beta S_n I_n \\ R_{n+1} = R_n - \gamma R_n + \nu I_n \end{cases}$ $R_n = N - S_n - I_n$

$N = 100,\ \nu = \gamma = 0.5 \Rightarrow$

$$\begin{cases} S_{n+1} = S_n - \beta S_n I_n + 0.5(100 - S_n - I_n) = 0.5 S_n - \beta S_n I_n + 50 - 0.5 I_n \\ I_{n+1} = I_n - 0.5 I_n + \beta S_n I_n = I_n(0.5 + \beta S_n) \end{cases}$$

Steady states

$$\begin{cases} \bar{S} = 0.5\bar{S} - \beta \bar{S}\bar{I} + 50 - 0.5\bar{I} \\ \bar{I} = \bar{I}(0.5 + \beta\bar{S}) \end{cases} \Leftrightarrow \begin{cases} \bar{S}(0.5 + \beta\bar{I}) = 50 - 0.5\bar{I} & (1) \\ \bar{I}(0.5 - \beta\bar{S}) = 0 & (2) \end{cases}$$

$(2) \Rightarrow \bar{I} = 0$ or $\bar{S} = \frac{0.5}{\beta}$

$\bar{I} = 0$ in (1) $\Rightarrow \bar{S} = 100$, $\bar{S} = \frac{0.5}{\beta}$ in (1) $\Rightarrow \bar{I} = 50 - \frac{1}{4\beta} \Rightarrow$

2 steady states $(\bar{S}_1, \bar{I}_1) = (100, 0)$ and $(\bar{S}_2, \bar{I}_2) = (\frac{1}{2\beta}, 50 - \frac{1}{4\beta})$

Note $(\bar{S}_2, \bar{I}_2)$ only exists with $\bar{I}_2 \geq 0$ if $50 - \frac{1}{4\beta} \geq 0 \Leftrightarrow \beta \geq \frac{1}{200}$

Stability

$$J(S, I) = \begin{pmatrix} 0.5 - \beta I & -\beta S - 0.5 \\ \beta I & 0.5 + \beta S \end{pmatrix} \Rightarrow$$

$$J(100, 0) = \begin{pmatrix} 0.5 & -100\beta - 0.5 \\ 0 & 0.5 + 100\beta \end{pmatrix}$$

$\Rightarrow \lambda_1 = 0.5,\ \lambda_2 = 0.5 + 100\beta$

$|\lambda_2| = 0.5 + 100\beta < 1$ if $\beta < \frac{1}{200}$

stable if $\beta < \frac{1}{200}$

$$J(\tfrac{1}{2\beta}, 50 - \tfrac{1}{4\beta}) = \begin{pmatrix} 0.75 - 50\beta & -1 \\ 50\beta - 0.25 & 1 \end{pmatrix} = J$$

Jury test (EK 2.8)

$|\underbrace{\mathrm{Tr}\, J}_{1.75 - 50\beta}| < 1 + \underbrace{\det J}_{0.5} < 2 \Leftrightarrow |\lambda_{1,2}| < 1$

$\Leftrightarrow |1.75 - 50\beta| < \underbrace{1.5 < 2}_{\text{OK}} \Leftrightarrow \frac{1}{200} < \beta < \frac{13}{200}$ condition for $(\bar{S}_2, \bar{I}_2)$ stable

$\beta = \frac{1}{250} \Rightarrow (100, 0)$ stable (only steady state)

$\beta = \frac{1}{100} \Rightarrow (100, 0)$ unstable and $(\bar{S}_2, \bar{I}_2) = (50, 25)$ stable

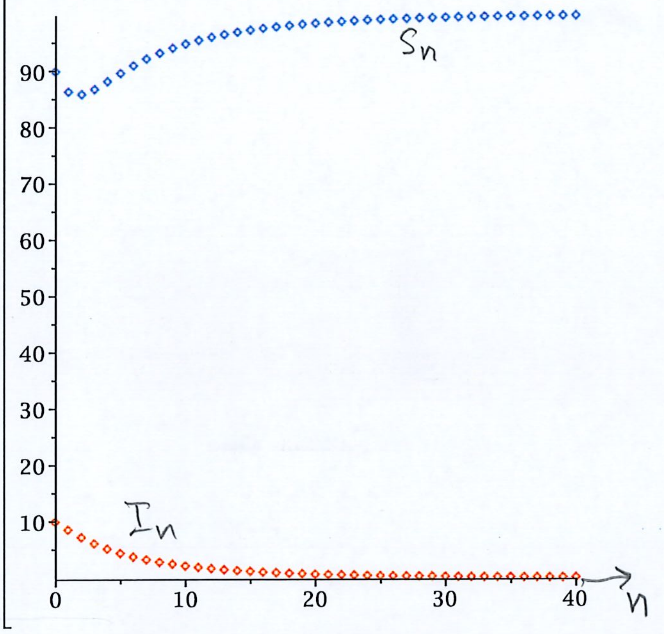
See Maple plots with $\beta = \frac{1}{250}$, $(S_n, I_n) \to (100, 0)$, $n \to \infty$

and $\beta = \frac{1}{100}$, $(S_n, I_n) \to (50, 25)$, $n \to \infty$.

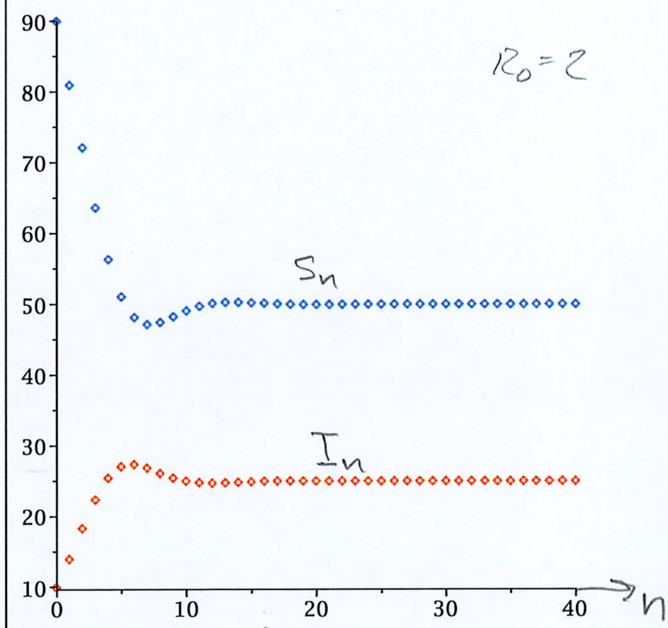
Exercise, discrete SIRS

$R_0 = 0.8$

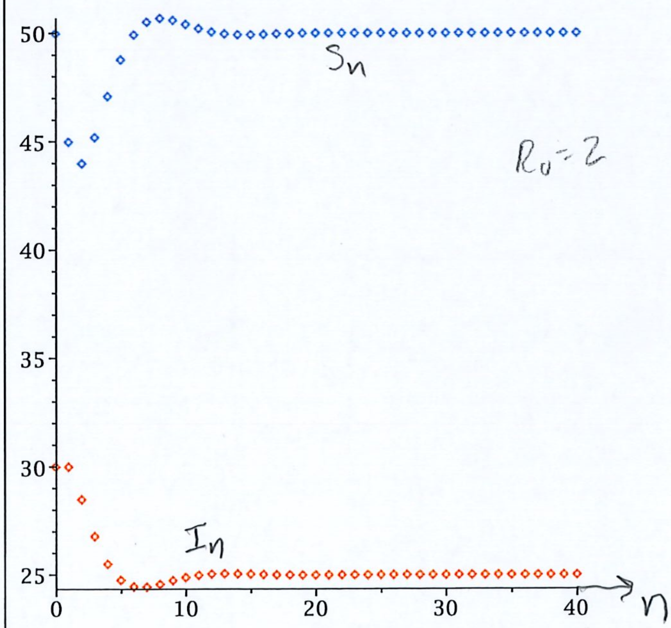
```
> s[0] := 90 : i[0] := 10 :
> for n from 0 to 39 do
    s[n+1] := s[n]·(0.5 − i[n]/250) + 50 − 0.5·i[n];
    i[n+1] := i[n]·(0.5 + s[n]/250);
  od:
> with(plots) :
> A := pointplot([seq([n, s[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, i[n]], n = 0..40)], color = red) :
> display(A, B)
```




```
> s[0] := 90 : i[0] := 10 :
> for n from 0 to 39 do
    s[n+1] := s[n]·(0.5 − i[n]/100) + 50 − 0.5·i[n];
    i[n+1] := i[n]·(0.5 + s[n]/100);
  od:
> A := pointplot([seq([n, s[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, i[n]], n = 0..40)], color = red) :
> display(A, B)
```




```
> s[0] := 50 : i[0] := 30 :
> for n from 0 to 39 do
    s[n+1] := s[n]·(0.5 − i[n]/100) + 50 − 0.5·i[n];
    i[n+1] := i[n]·(0.5 + s[n]/100);
  od:
> A := pointplot([seq([n, s[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, i[n]], n = 0..40)], color = red) :
> display(A, B)
```

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(7) $x_{n+1} = f(x_n) = -x_n^2(1-x_n)$

Steady states:

$\bar{x} = f(\bar{x}) = -\bar{x}^2(1-\bar{x}) \iff \bar{x}(1+\bar{x}-\bar{x}^2) = 0 \Rightarrow$

$\bar{x}_1 = 0$, $\bar{x}_{2,3} = \frac{1 \pm \sqrt{5}}{2}$ ($\bar{x}_2$ is the golden mean), $\bar{x}_2 \approx 1.62$, $\bar{x}_3 \approx -0.38$

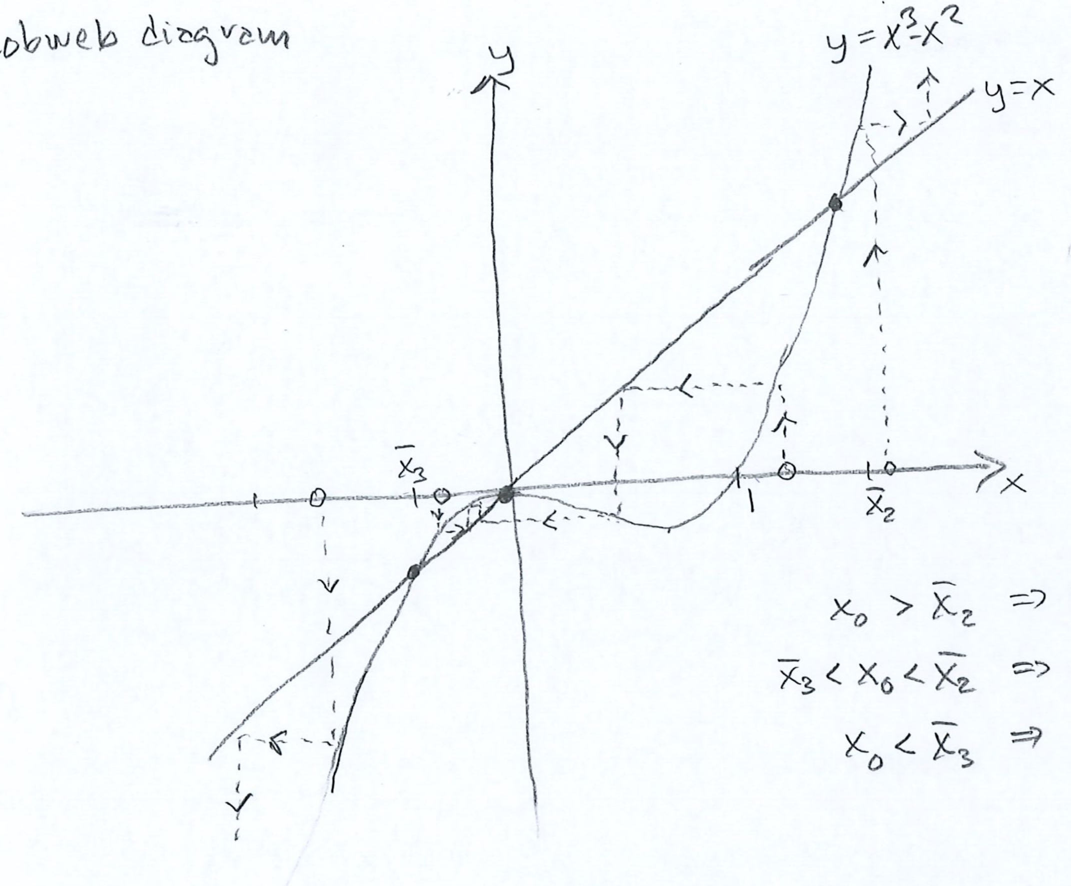
Stable?

$f(x) = -x^2 + x^3 \Rightarrow f'(x) = -2x + 3x^2 \Rightarrow$

$|f'(\bar{x}_1)| = |f'(0)| = 0 < 1 \Rightarrow$ stable

$|f'(\bar{x}_{2,3})| = \left|3\left(\frac{1\pm\sqrt{5}}{2}\right)^2 - 2\cdot\frac{1\pm\sqrt{5}}{2}\right| = \left|\frac{3}{4}(1 \pm 2\sqrt{5} + 5) - (1 \pm \sqrt{5})\right| = \frac{7 \pm \sqrt{5}}{2} > 1 \Rightarrow \bar{x}_2$ and $\bar{x}_3$ unstable

Cobweb diagram



circle o on x-axis = some initial value x_0

$x_0 > \bar{x}_2 \Rightarrow x_n \to \infty,\ n \to \infty$

$\bar{x}_3 < x_0 < \bar{x}_2 \Rightarrow x_n \to 0 = \bar{x}_1,\ n \to \infty$

$x_0 < \bar{x}_3 \Rightarrow x_n \to -\infty,\ n \to \infty$

(8) $N_{n+1} = f(N_n) = \dfrac{\lambda N_n}{(1+N_n)^2}$, $\lambda > 0$, $N_n \geq 0$

Steady states

$$\bar{N} = \frac{\lambda \bar{N}}{(1+\bar{N})^2} \Rightarrow \bar{N}_1 = 0 \text{ or } \frac{\lambda}{(1+\bar{N}_2)^2} = 1 \Rightarrow \underset{(\bar{N}_2 \geq 0)}{1+\bar{N}_2 = \overset{+}{\underset{(-)}{}}\sqrt{\lambda}} \Rightarrow \bar{N}_2 = \sqrt{\lambda} - 1 \geq 0 \text{ if } \lambda \geq 1$$

Stable?

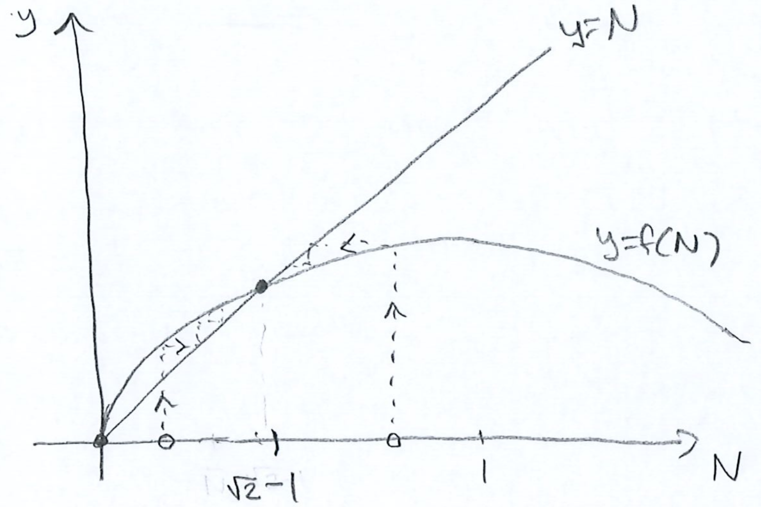
$$f(N) = \frac{\lambda N}{(1+N)^2} \Rightarrow f'(N) = \frac{\lambda}{(1+N)^2} - \frac{2\lambda N}{(1+N)^3} = \frac{\lambda(1+N) - 2\lambda N}{(1+N)^3} = \frac{\lambda(1-N)}{(1+N)^3} \Rightarrow$$

$|f'(\bar{N}_1)| = |f'(0)| = \lambda$, stable if $\lambda < 1$ (i.e., when $\bar{N}_2$ does not exist positive)

$|f'(\bar{N}_2)| = \left|\dfrac{\lambda(2-\sqrt{\lambda})}{\lambda^{3/2}}\right| = \left|\dfrac{2}{\sqrt{\lambda}} - 1\right| < 1$ (stable) if $0 < \dfrac{2}{\sqrt{\lambda}} < 2$, true if $\lambda > 1$

So , for $\lambda < 1$ $\bar{N}_1 = \bar{0}$ is stable , for $\lambda > 1$ $\bar{N}_2 = \sqrt{\lambda} - 1$ is stable and $\bar{N}_1 = 0$ unstable

Cobweb for $\lambda = 2$



$\bar{N}_2 = \sqrt{2} - 1 \approx 0.4$ stable

$\bar{N}_1 = 0$ unstable

$f(N) = \dfrac{2N}{(1+N)^2} \to 0 \quad N \to \infty$

$f'(N) = \dfrac{2(1-N)}{(1+N)^3} \begin{cases} > 0, & N < 1 \\ = 0, & N = 1 \\ < 0, & N > 1 \end{cases}$

$\Rightarrow f$ has a max $f(1) = \frac{1}{2}$

$N_n \to \sqrt{2} - 1 = \bar{N}_2$, $n \to \infty$ if $N_0 > 0$

(9) The system is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 3/2 & 3/2 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}}_{L} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

L is a Leslie matrix

Eigenvalues: $\det(L-\lambda I) = \begin{vmatrix} -\lambda & 3/2 & 3/2 \\ 1/2 & -\lambda & 0 \\ 0 & 1/3 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4} = 0$

Observe that $\lambda_1 = 1$ is a solution. Polynomial division $\Rightarrow$

$$-\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4} = -(\lambda-1)(\lambda^2+\lambda+\frac{1}{4}) = -(\lambda-1)(\lambda+\frac{1}{2})^2$$

$\Rightarrow \lambda_{2,3} = -1/2 \qquad |\lambda_{2,3}| < \lambda_1 = 1$

For n large, $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \approx c \cdot \lambda_1^n \bar{v}_1$, $\bar{v}_1$ is the eigenvector of $\lambda_1 = 1$:

$(L-I)\bar{v}_1 = \bar{0}$: $\left(\begin{array}{ccc|c} -1 & 3/2 & 3/2 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1/3 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1/3 & -1 & 0 \end{array}\right) \Rightarrow \bar{v}_1 = t\begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$

Choose $t = 0.1$, $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \approx c_1 \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix}$

60% in youngest class
30% in middle
10% in oldest

Remarks: L is a non-negative irreducible matrix
The eigenvalue $\lambda_1 = 1$ is positive and real, and $\lambda_1 > |\lambda_j|$ for $j = 2, 3$.
The eigenvector $\begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$ of λ_1 is positive ($\lambda_2 = \lambda_3$ will not have any positive eigenvector)
All this in agreement with Frobenius theorem.