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TATM38, Exercises on PDE models

IBVP's

Solve the following initial-boundary value problems

1.
$$\begin{cases} u_t = u_{xx} \ , \quad 0 < x < 1, \ t > 0 \\ u(t,0) = 0 \ , \quad t > 0 \\ u(t,1) = 0 \ , \quad t > 0 \\ u(0,x) = x(1-x) \ , \quad 0 < x < 1 \end{cases}$$

Give an approximate expression for $u(t,x)$ for large t.

2.
$$\begin{cases} u_t = u_{xx} \ , \quad 0 < x < \pi, \ t > 0 \\ u_x(t,0) = 0 \ , \quad t > 0 \\ u_x(t,\pi) = 0 \ , \quad t > 0 \\ u(0,x) = f(x) \ , \quad 0 < x < \pi \end{cases}$$

with a) $f(x) = 7\cos 2x$, and b) $f(x) = \pi - x$

3.
$$\begin{cases} u_t = 3u_{xx} \ , \quad 0 < x < 2, \ t > 0 \\ u(t,0) = 1 \ , \quad t > 0 \\ u(t,2) = 3 \ , \quad t > 0 \\ u(0,x) = 1 + x + \sin(\pi x) \ , \quad 0 < x < 2 \end{cases}$$

Hint: Put $u(t,x) = v(t,x) + w(x)$ and determine $w(x)$ such that an IBVP with homogeneous boundary conditions is obtained for $v(t,x)$.

4.
$$\begin{cases} 4u_t = u_{xx} - 2u \ , \quad 0 < x < 2\pi, \ t > 0 \\ u_x(t,0) = 0 \ , \quad t > 0 \\ u_x(t,2\pi) = 0 \ , \quad t > 0 \\ u(0,x) = x \ , \quad 0 < x < 2\pi \end{cases}$$

Hint: To remove the term $-2u$, put $u(t,x) = v(t,x)e^{\alpha t}$ and determine the right value of α. State the IBVP obtained for $v(t,x)$ and solve it first.

5.
$$\begin{cases} u_t = 3u_{xx} - 4u_x \ , \quad 0 < x < \pi, \ t > 0 \\ u(t,0) = 0 \ , \quad t > 0 \\ u(t,\pi) = 0 \ , \quad t > 0 \\ u(0,x) = e^{2x/3}\sin 4x \ , \quad 0 < x < \pi \end{cases}$$

Hint: To remove the term $-4u_x$, put $u(t,x) = v(t,x)e^{\alpha t + \beta x}$ and determine α and β. State the IBVP obtained for $v(t,x)$ and solve it first.

6.
$$\begin{cases} u_t = u_{xx} + u_{yy} \ , \quad 0 < x < \pi, \ 0 < y < \pi, \ t > 0 \\ u_x(t,0,y) = 0 \ , \quad 0 < y < \pi, \ t > 0 \\ u_x(t,\pi,y) = 0 \ , \quad 0 < y < \pi, \ t > 0 \\ u_y(t,x,0) = 0 \ , \quad 0 < x < \pi, \ t > 0 \\ u_y(t,x,\pi) = 0 \ , \quad 0 < x < \pi, \ t > 0 \\ u(0,x,y) = 4\cos x\cos 3y + 5\cos 2x\cos 2y \ , \quad 0 < x < \pi, \ 0 < y < \pi \end{cases}$$

Turing pattern formation

7. [EK 11.15a] Show that Turing diffusive instabilitiy cannot occur for the Lotka-Volterra predator-prey equations with spatial diffusion, here $u(t,x) =$ prey and $v(t,x) =$ predators :

$$\begin{cases} u_t = au - buv + D_1 u_{xx} \\ v_t = -cv + duv + D_2 v_{xx} \end{cases}$$

8. [EK 11.15c] Can one have diffusive instabilitiy for the Van der Pol oscillator with diffusion:

$$\begin{cases} u_t = u - \frac{1}{3}u^3 + v + D_1 u_{xx} \\ v_t = -u + D_2 v_{xx} \end{cases} \quad ?$$

9. [$\sim$ EK 11.15f] Formulate the conditions on the parameters α_1, α_2, α_3, α_4 and α_5, all $\alpha_j > 0$, and on D_1 and D_2, for the existence of a stable steady state in absence of diffusion ($D_1 = D_2 = 0$) with diffusive instabilities, for the phytoplankton-herbivore system

$$\begin{cases} u_t = \alpha_1 u + \alpha_2 u^2 - \alpha_3 uv + D_1 u_{xx} \\ v_t = -\alpha_4 v^2 + \alpha_5 uv + D_2 v_{xx} \end{cases}$$

10. [$\sim$ EK 11.15h] Find the (positive) spatially uniform steady state of the Meinhardt model with diffusion ($\alpha > 0$, $\beta > 0$ and $\mu > 0$ constants) :

$$\begin{cases} u_t = \alpha u^2 v - \mu u + D_1 u_{xx} \\ v_t = \beta - \alpha u^2 v + D_2 v_{xx} \end{cases}$$

What is the condition on α, β and μ for the steady state to be stable in absence of diffusion?
What is the condition on α, β, μ, D_1 and D_2 for Turing diffusive instability?
Choose $\alpha = 2$, $\beta = \mu = 1$ and $D_1 = 1$. For what values of D_2 can we have diffusive instability?
If $D_2 = 12$ and $0 < x < 25$, what patterns can form?

In two space dimensions, with $\alpha = 2$, $\beta = \mu = 1$, $D_1 = 1$, $D_2 = 12$, $0 < x < 25$ and $0 < y < 25$, what patterns can form from Turing instability in the Meinhardt system

$$\begin{cases} u_t = \alpha u^2 v - \mu u + D_1 (u_{xx} + u_{yy}) \\ v_t = \beta - \alpha u^2 v + D_2 (v_{xx} + v_{yy}) \end{cases} \quad ?$$

Solutions next pages

①
$$\begin{cases} u_t = u_{xx} & t>0,\ 0<x<1 \quad (1)\\ u(t,0)=0 & \} t>0 \quad (2) \\ u(t,1)=0 & \\ u(0,x)=x(1-x), & 0<x<1 \quad (3) \end{cases}$$

$$u(t,x)=T(t)X(x) \Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda \Rightarrow T(t)=ke^{\lambda t}$$

$(2) \Rightarrow X(0)=X(1)=0$

$$\left.\begin{aligned} X''(x)-\lambda X(x)&=0\\ X(0)=X(1)&=0\end{aligned}\right\} \Rightarrow X_n(x)=d_n\sin(n\pi x) \text{ and } \lambda=-n^2\pi^2 \text{ only non-zero solutions}$$

(1) linear, (2) homogeneous $\Rightarrow$

$u(t,x)=\sum_{n=1}^{\infty}\alpha_n e^{-n^2\pi^2 t}\sin(n\pi x)$ solves (1) and (2)

$(3) \Rightarrow u(0,x)=\sum_{n=1}^{\infty}\alpha_n\sin(n\pi x)=x(1-x)$, sin-series of $x(1-x) \Rightarrow$

$$\alpha_n = 2\int_0^1 x(1-x)\sin n\pi x\,dx = 2\left[(x-x^2)\frac{-\cos n\pi x}{n\pi}\right]_0^1 - 2\int_0^1(1-2x)\frac{-\cos n\pi x}{n\pi}\,dx =$$

$$= 0 - 2\left[(1-2x)\frac{-\sin n\pi x}{n^2\pi^2}\right]_0^1 + 2\int_0^1(-2)\frac{-\sin n\pi x}{n^2\pi^2}\,dx = 0+\left[\frac{-4\cos n\pi x}{n^3\pi^3}\right]_0^1 =$$

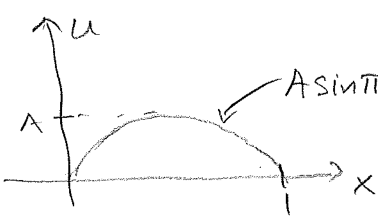
$$= \frac{4}{n^3\pi^3}\Big(-\underbrace{\cos(n\pi)}_{(-1)^n}+1\Big) = \begin{cases}\frac{8}{n^3\pi^3}, & n \text{ odd}\\ 0, & n \text{ even}\end{cases} \Rightarrow$$

$$u(t,x)=\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}e^{-n^2\pi^2 t}\sin(n\pi x) =$$

$$= \frac{8}{\pi^3}\left(e^{-\pi^2 t}\sin(\pi x)+\frac{1}{27}e^{-9\pi^2 t}\sin(3\pi x)+\frac{1}{125}e^{-25\pi^2 t}\sin(5\pi x)+\dots\right)$$

For t large,

$u(t,x) \approx \frac{8}{\pi^3}e^{-\pi^2 t}\sin(\pi x)$ (the other terms $\to 0$ faster)



$A\sin\pi x$, amplitude A decreases exponentially with time

② $\begin{cases} u_t = u_{xx}, & 0<x<\pi,\ t>0 \quad (1) \\ u_x(t,0)=0, & t>0 \\ u_x(t,\pi)=0, & t>0 \end{cases}$ (2)

$u(0,x)=f(x)$, $0<x<\pi$ (3)

$u(t,x) = T(t)X(x) \underset{(1)}{\Rightarrow} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda \Rightarrow T(t) = ke^{\lambda t}$

$(2) \Rightarrow X'(0) = X'(\pi) = 0$

$\left.\begin{array}{l} X''(x) - \lambda X(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{array}\right\}$ $\begin{array}{l} \lambda>0 \Rightarrow X(x)=0 \text{ only} \\ \lambda = 0 \Rightarrow X(x) = a = \text{constant is a solution} \\ \lambda<0 \Rightarrow X_n(x) = a_n \cos nx \text{ are solutions}, \lambda=-n^2 \end{array}$

$\Rightarrow u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos nx$ solves (1) and (2)

$(3) \Rightarrow u(0,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = f(x)$, cos-series of $f(x)$ on $(0,\pi)$

a) $f(x) = 7\cos 2x \Rightarrow$ identify $a_2 = 7$, $a_n = 0, n \neq 2 \Rightarrow$

$\underline{u(t,x) = 7e^{-4t}\cos 2x}$

b) $f(x) = \pi - x \Rightarrow$

$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx \underset{n\neq 0}{=} \frac{2}{\pi}\underbrace{\left[(\pi-x)\frac{\sin nx}{n}\right]_0^{\pi}}_{=0} - \frac{2}{\pi}\int_0^{\pi}(-1)\frac{\sin nx}{n}\,dx =$

$= -\frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi n^2}(1-\cos n\pi) = \frac{2}{\pi n^2}(1-(-1)^n) = \begin{cases} \frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cdot 1\,dx = \frac{2}{\pi}\left[-\frac{(\pi-x)^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$

$\Rightarrow \underline{u(t,x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} e^{-n^2 t}\cos nx}$

(3) $\begin{cases} u_t = 3u_{xx} & ,0<x<2, t>0 \quad (1) \\ u(t,0)=1 & , t>0 \\ u(t,2)=3 & , t>0 \end{cases} (2)$
$u(0,x) = 1+x+\sin\pi x \quad (3)$

Remove inhomogenities in (2). Try $u(t,x) = v(t,x) + w(x) \Rightarrow$

$u_t = v_t$, $u_x = v_x + w'(x)$, $u_{xx} = v_{xx} + w''(x)$

$$\begin{aligned} (1) &\Rightarrow & v_t &= 3v_{xx} + 3w''(x) \\ (2) &\Rightarrow & v(t,0) &= u(t,0) - w(0) = 1 - w(0) \\ & & v(t,2) &= u(t,2) - w(2) = 3 - w(2) \\ 3 &\Rightarrow & v(0,x) &= u(0,x) - w(x) = 1 + x + \sin\pi x - w(x) \end{aligned}$$

Would like $\begin{cases} w''(x) = 0 \\ w(0) = 1 \\ w(2) = 3 \end{cases}$ $\Rightarrow w(x) = ax+b \Rightarrow w(0) = b = 1$

$w(2) = 2a+b = 3 \Rightarrow a = 1$

$\Rightarrow w(x) = x+1$

IBVP for $v(t,x)$:

$\begin{cases} v_t = 3v_{xx} & , (4) \\ v(t,0) = 0 \\ v(t,2) = 0 \end{cases} (5)$
$v(0,x) = \sin\pi x \quad (6)$

$v(t,x) = T(t)X(x) \Rightarrow \dfrac{T'(t)}{3T(t)} = \dfrac{X''(x)}{X(x)} = \lambda$

$\Rightarrow T(t) = k\cdot e^{3\lambda t}$

$(5) \Rightarrow X(0) = X(2) = 0$

$\left.\begin{matrix} X''(x) - \lambda X(x) = 0 \\ X(0) = X(2) = 0 \end{matrix}\right\}$

$\lambda > 0 \Rightarrow X(x) = 0$ only
$\lambda = 0 \Rightarrow X(x) = 0$ only
$\lambda < 0 \Rightarrow X_n(x) = a_n \sin\frac{n\pi x}{2}, n=1,2,\dots \left(\lambda = -\frac{n^2\pi^2}{4}\right)$

Solutions to (4) and (5) are

$$v(t,x) = \sum_{n=1}^{\infty} a_n e^{-3n^2\pi^2 t/4} \sin\frac{n\pi x}{2}$$

$(6) \Rightarrow \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{2} = \sin\pi x$, identify $a_n = \begin{cases} 1 & \text{if } n=2 \\ 0 & \text{if } n \neq 2 \end{cases}$

$\Rightarrow v(t,x) = e^{-3\pi^2 t} \sin\pi x$

$\Rightarrow u(t,x) = 1 + x + e^{-3\pi^2 t}\sin\pi x$

(4)
$$\begin{cases} 4u_t = u_{xx} - 2u & , 0<x<2\pi, t>0 \quad (1) \\ u_x(t,0)=0 & , t>0 \\ u_x(t,2\pi)=0 & , t>0 \end{cases} \Big\} (2)$$
$$u(0,x) = x \quad , 0<x<2\pi \quad (3)$$

$u(t,x) = v(t,x)e^{\alpha t} \Rightarrow u_t = v_t e^{\alpha t} + \alpha v e^{\alpha t} = (v_t + \alpha v)e^{\alpha t}$,

$u_x = v_x e^{\alpha t}$, $u_{xx} = v_{xx} e^{\alpha t}$ $\quad (1) \Rightarrow 4(v_t + \alpha v)e^{\alpha t} = v_{xx}e^{\alpha t} - 2ve^{\alpha t} \Rightarrow$

$4v_t + 4\alpha v = v_{xx} - 2v$. Take $\alpha = -\frac{1}{2}$ $\Rightarrow 4v_t = v_{xx}$ and $u = ve^{-t/2} \Leftrightarrow v = ue^{t/2}$

IBVP for $v(t,x)$

$$\begin{cases} 4v_t = v_{xx} & , 0<x<2\pi, t>0 \quad (4) \\ v_x(t,0) = u_x(t,0)e^{t/2} = 0 & , t>0 \\ v_x(t,2\pi) = u_x(t,2\pi)e^{t/2} = 0 & , t>0 \quad (5) \\ v(0,x) = u(0,x)e^{0/2} = x & , 0<x<2\pi \end{cases}$$

$v(t,x) = T(t)X(x) \Rightarrow 4\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda \Rightarrow T'(t) = \frac{\lambda}{4}T(t) \Rightarrow T(t) = ke^{\lambda t/4}$

$(5) \Rightarrow X'(0) = X'(2\pi) = 0$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X'(0) = X'(2\pi) = 0 \end{cases} \Rightarrow \begin{array}{l} \lambda > 0 : \text{only } X(x) = 0 \\ \lambda = 0 : X(x) = a = \text{constant} \\ \lambda < 0 : X_n(x) = a_n \cos\frac{nx}{2}, \ \lambda = -\frac{n^2}{4} \end{array}$$

(4) linear, (5) homogeneous $\Rightarrow$ (4) and (5) are solved by

$$v(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t/16} \cos\frac{nx}{2} \qquad (a_0, a_1, \dots \text{ any constants})$$

$(6) \Rightarrow v(0,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{nx}{2}$, cos-series of x on $(0,2\pi) \Rightarrow$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} x\cos\frac{nx}{2}dx \underset{n\neq 0}{=} \frac{1}{\pi}\underbrace{\left[x\frac{\sin\frac{nx}{2}}{n/2}\right]_0^{2\pi}}_{=0} - \frac{1}{\pi}\int_0^{2\pi} 1\cdot\frac{\sin\frac{nx}{2}}{n/2}dx = \frac{1}{\pi}\left[\frac{\cos\frac{nx}{2}}{(n/2)^2}\right]_0^{2\pi} =$$

$$= \frac{4}{\pi n^2}(\cos n\pi - 1) = \frac{4}{\pi n^2}((-1)^n - 1) = \begin{cases} -\frac{8}{\pi n^2} & , n \text{ odd} \\ 0 & , n \text{ even} \end{cases}$$

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} x \cdot 1\, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$\Rightarrow v(t,x) = \pi + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}e^{-n^2 t/16}\cos\frac{nx}{2} \Rightarrow$$

$$u(t,x) = v(t,x)e^{-t/2} = \pi e^{-t/2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}e^{-(n^2+8)t/16}\cos\frac{nx}{2} =$$

$$= \pi e^{-t/2} - \frac{8}{\pi}e^{-9t/16}\cos\frac{x}{2} - \frac{8}{9\pi}e^{-17t/16}\cos\frac{3x}{2} - \dots$$

(5) $\begin{cases} u_t = 3u_{xx} - 4u_x & , 0<x<\pi, t>0 \quad (1) \\ u(t,0)=0 & , t>0 \\ u(t,\pi)=0 & , t>0 \end{cases} \Big\} (2)$

$\quad u(0,x) = e^{2x/3}\sin 4x \quad , 0<x<\pi \quad (3)$

$u(t,x) = v(t,x)e^{\alpha t+\beta x} \Rightarrow u_t = (v_t+\alpha v)e^{\alpha t+\beta x}, \; u_x = (v_x+\beta v)e^{\alpha t+\beta x},$

$u_{xx} = (v_{xx}+\beta v_x + \beta(v_x+\beta v))e^{\alpha t+\beta x}, \quad (1) \Rightarrow$

$(v_t+\alpha v)e^{\alpha t+\beta x} = 3(v_{xx}+2\beta v_x+\beta^2 v)e^{\alpha t+\beta x} - 4(v_x+\beta v)e^{\alpha t+\beta x} \Rightarrow$

$v_t + \alpha v = 3v_{xx} + (6\beta-4)v_x + (3\beta^2-4\beta)v \Rightarrow v_t = 3v_{xx} + (6\beta-4)v_x + (3\beta^2-4\beta-\alpha)v$

Choose α, β such that $6\beta-4=0$ and $3\beta^2-4\beta-\alpha=0 \Rightarrow \beta = 2/3, \alpha = -4/3 \Rightarrow$

$u(t,x) = v(t,x)e^{-4t/3+2x/3} \Leftrightarrow v(t,x) = u(t,x)e^{4t/3-2x/3}$

IBVP for $v(t,x)$

$\begin{cases} v_t = 3v_{xx} & , 0<x<\pi, t>0 \quad (4) \\ v(t,0) = u(t,0)e^{4t/3-0} = 0 & , t>0 \\ v(t,\pi) = u(t,\pi)e^{4t/3-2\pi/3} = 0 & , t>0 \end{cases} \Big\} (5)$

$\quad v(0,x) = u(0,x)e^{0-2x/3} = e^{2x/3}\sin 4x \, e^{-2x/3} = \sin 4x, 0<x<\pi \quad (6)$

$v(t,x) = T(t)X(x) \Rightarrow \dfrac{T'(t)}{3T(t)} = \dfrac{X''(x)}{X(x)} = \lambda \quad \Rightarrow T(t) = k\cdot e^{3\lambda t}$

$(5) \Rightarrow X(0) = X(\pi) = 0$

$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = X(\pi) = 0 \end{cases} \Rightarrow \quad \begin{array}{l} \lambda \geq 0: \text{ only } X(x)=0 \\ \lambda<0: X_n(x) = d_n \sin nx, n=1,2,\dots, \lambda = -n^2 \end{array}$

$\Rightarrow \quad v(t,x) = \sum_{n=1}^{\infty} \alpha_n e^{-3n^2 t}\sin nx$ solves (4) and (5)

(6) $v(0,x) = \sum_{n=1}^{\infty} \alpha_n \sin nx = \sin 4x$, identify $\alpha_4 = 1, \alpha_n = 0, n\neq 4$

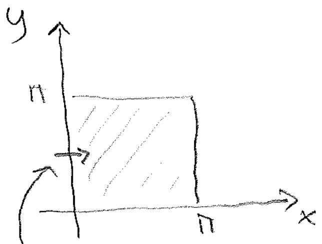
$\Rightarrow \quad v(t,x) = e^{-48t}\sin 4x \quad \Rightarrow$

$$u(t,x) = v(t,x)e^{-4t/3+2x/3} = e^{-148t/3+2x/3}\sin 4x$$

[Exercise: calculate u_t, u_x and u_{xx} and verify that it satisfies (1), also verify (2) and (3)]

⑥
$$\begin{cases} u_t = u_{xx} + u_{yy} & , t>0, 0<x<\pi, 0<y<\pi \quad (1) \\ u_x(t,0,y) = 0 & , t>0, 0<y<\pi \\ u_x(t,\pi,y) = 0 & , t>0, 0<y<\pi \\ u_y(t,x,0) = 0 & , t>0, 0<x<\pi \\ u_y(t,x,\pi) = 0 & , t>0, 0<x<\pi \\ u(0,x,y) = 4\cos x\cos 3y + 5\cos 2x \cos 2y & (3) \ \text{(IC)} \end{cases}$$

(2) (BC) for the four boundary conditions $u_x(t,0,y)=0$ … $u_y(t,x,\pi)=0$



$u_x(t,0,y)=0$ means no flow in or out through boundary on the left

Separation of variables

$u(t,x,y) = T(t)X(x)Y(y)$ into (1) $\Rightarrow$

$$\frac{T'(t)}{T(t)} = \underbrace{\frac{X''(x)}{X(x)}}_{=\lambda} + \underbrace{\frac{Y''(y)}{Y(y)}}_{=\mu} \Rightarrow T'(t) = (\lambda+\mu)T(t) \Rightarrow T(t) = k \cdot e^{(\lambda+\mu)t}$$

$$(2) \Rightarrow X'(0) = X'(\pi) = Y'(0) = Y'(\pi) = 0$$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X'(0) = X'(\pi) = 0 \end{cases} \Rightarrow \begin{array}{l} \lambda > 0 : \text{only } X(x) = 0 \\ \lambda = 0 : \ X(x) = a = \text{constant} \\ \lambda < 0 : \ X_n(x) = a_n \cos nx, \ n = 1, 2, \ldots, \ \lambda = -n^2 \end{array}$$

$$\begin{cases} Y''(y) - \mu Y(y) = 0 \\ Y'(0) = Y'(\pi) = 0 \end{cases} \xRightarrow[\text{same way}]{\text{in the}} Y_m(y) = b_m \cos my, \ m = 0, 1, 2, \ldots, \ \mu = -m^2$$

$m=0$ when $\mu = 0$

(1) linear, (2) homogeneous $\Rightarrow$

$$u(t,x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{n,m} e^{-(n^2+m^2)t} \cos nx \cos my$$

solves (1) and (2) for all constants $\alpha_{n,m}$

$$(3) \Rightarrow u(0,x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{n,m} \cos nx \cos my = 4\cos x\cos 3y + 5\cos 2x\cos 2y$$

$$\Rightarrow \alpha_{1,3} = 4, \ \alpha_{2,2} = 5, \ \text{all other } \alpha_{n,m} = 0 \Rightarrow$$

$$\underline{u(t,x,y) = 4e^{-10t}\cos x\cos 3y + 5e^{-8t}\cos 2x\cos 2y}$$

(7) $\begin{cases} u_t = au - buv + D_1 u_{xx} \\ v_t = -cv + duv + D_2 u_{yy} \end{cases}$, $R_1(u,v) = au - buv$, $R_2(u,v) = -cv + duv$

Spatially uniform steady states

$\begin{cases} \bar{u}(a - b\bar{v}) = 0 \\ \bar{v}(-c + d\bar{u}) = 0 \end{cases}$, $\bar{u} = 0 \Downarrow \bar{v} = 0$ or $\bar{v} = a/b \Downarrow \bar{u} = c/d$ $\quad (0,0)$ and $(\frac{c}{d}, \frac{a}{b})$

$$A(u,v) = \begin{pmatrix} \frac{\partial R_1}{\partial u} & \frac{\partial R_1}{\partial v} \\ \frac{\partial R_2}{\partial u} & \frac{\partial R_2}{\partial v} \end{pmatrix} = \begin{pmatrix} a - bv & -bu \\ dv & -c + du \end{pmatrix} \Rightarrow$$

$A(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ $\lambda_1 = a > 0$ $(\lambda_2 = -c)$ $\Rightarrow$ unstable $\Rightarrow$ no diffusive instability possible

$A(\frac{c}{d}, \frac{a}{b}) = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix} = A$ $\quad \begin{cases} \mathrm{Tr}\, A = 0 & \text{limit case} \\ \det A = ac > 0 & \text{ok} \end{cases}$

Condition $\underbrace{a_{11}}_{=0} D_2 + \underbrace{a_{22}}_{=0} D_1 > 2\sqrt{D_1 D_2 \det A} \iff 0 > 2\underbrace{\sqrt{D_1 D_2 ac}}_{>0}$ cannot be satisfied

$\Rightarrow$ diffusive instability impossible

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(8) $\begin{cases} u_t = u - \frac{u^3}{3} + v + D_1 u_{xx} \\ v_t = -u + D_2 v_{xx} \end{cases}$

Spatially uniform steady states: $\left.\begin{array}{l} \bar{u} - \frac{\bar{u}^3}{3} + \bar{v} = 0 \\ -\bar{u} = 0 \end{array}\right\} \Rightarrow \bar{u} = \bar{v} = 0$

$$A(u,v) = \begin{pmatrix} 1 - u^2 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow A = A(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$\mathrm{Tr}\, A > 0 \Rightarrow (\bar{u}, \bar{v}) = (0,0)$ unstable $\Rightarrow$ no point to check for diffusive instability

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(9) $$\begin{cases} u_t = \alpha_1 u + \alpha_2 u^2 - \alpha_3 uv + D_1 u_{xx} \\ v_t = -\alpha_4 v^2 + \alpha_5 uv + D_2 v_{xx} \end{cases}$$

Spatially uniform steady states:

$$\begin{cases} \bar u(\alpha_1 + \alpha_2 \bar u - \alpha_3 \bar v) = 0 \quad (1) \\ \bar v(-\alpha_4 \bar v + \alpha_5 \bar u) = 0 \quad (2) \Rightarrow \bar v = 0 \text{ or } \bar v = \frac{\alpha_5}{\alpha_4}\bar u \end{cases}$$

$\bar v = 0$ in (1) $\Rightarrow \bar u = 0$ (or $\bar u = -\frac{\alpha_1}{\alpha_2} < 0$)

$\bar v = \frac{\alpha_5}{\alpha_4}\bar u$ in (1) $\Rightarrow \bar u\Big(\alpha_1 + \underbrace{(\alpha_2 - \frac{\alpha_3\alpha_5}{\alpha_4})}_{\frac{\alpha_2\alpha_4 - \alpha_3\alpha_5}{\alpha_4}}\bar u\Big) = 0 \Rightarrow \bar u = 0$ or $\bar u = \frac{\alpha_1\alpha_4}{\alpha_3\alpha_5 - \alpha_2\alpha_4}$

$\Rightarrow (0,0)$ and $\left(\frac{\alpha_1\alpha_4}{\alpha_3\alpha_5 - \alpha_2\alpha_4}, \frac{\alpha_1\alpha_5}{\alpha_3\alpha_5 - \alpha_2\alpha_4}\right) = (\bar u_1, \bar v_1)$ steady states, if $\underline{\alpha_3\alpha_5 > \alpha_2\alpha_4}$

$$A(u,v) = \begin{pmatrix} \alpha_1 + 2\alpha_2 u - \alpha_3 v & -\alpha_3 u \\ \alpha_5 v & -2\alpha_4 v + \alpha_5 u \end{pmatrix} \Rightarrow$$

$$A(0,0) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda_1 = \alpha_1 > 0 \Rightarrow \text{unstable}$$

$$A(\bar u_1, \bar v_1) = \begin{pmatrix} \alpha_2 \bar u_1 & -\alpha_3 \bar u_1 \\ \alpha_5 \bar v_1 & -\alpha_4 \bar v_1 \end{pmatrix} = \underbrace{\frac{\alpha_1}{\alpha_3\alpha_5 - \alpha_2\alpha_4}}_{=k} \begin{pmatrix} \alpha_2\alpha_4 & -\alpha_3\alpha_4 \\ \alpha_5^2 & -\alpha_4\alpha_5 \end{pmatrix} = A$$

1. $\det A = k^2(-\alpha_2\alpha_4^2\alpha_5 + \alpha_3\alpha_4\alpha_5^2) = k^2\alpha_4\alpha_5(\alpha_3\alpha_5 - \alpha_2\alpha_4) > 0$ if $(\bar u_1, \bar v_1)$ exists
2. $\operatorname{Tr} A = k\cdot(\alpha_2\alpha_4 - \alpha_4\alpha_5) = k\alpha_4(\alpha_2 - \alpha_5) < 0$ if $\underline{\alpha_2 < \alpha_5}$

$\Rightarrow (\bar u_1, \bar v_1)$ stable for $D_1 = D_2 = 0$ if $\underline{\alpha_2 < \alpha_5}$ (and $\underline{\alpha_3\alpha_5 > \alpha_2\alpha_4}$ needed for $(\bar u_1, \bar v_1)$ positive)

3. Diffusive instability if $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A} \iff$

$k(\alpha_2\alpha_4 D_2 - \alpha_4\alpha_5 D_1) > 2\sqrt{D_1 D_2 k^2 \alpha_4 \alpha_5(\alpha_3\alpha_5 - \alpha_2\alpha_4)} \iff$

$\sqrt{\alpha_4}(\alpha_2 D_2 - \alpha_5 D_1) > 2\sqrt{D_1 D_2 \alpha_5(\alpha_3\alpha_5 - \alpha_2\alpha_4)} \iff$

$$\underline{\frac{D_2}{D_1} - \frac{\alpha_5}{\alpha_2} > \frac{2}{\alpha_2}\sqrt{\frac{D_2}{D_1}\cdot\frac{\alpha_5}{\alpha_4}(\alpha_3\alpha_5 - \alpha_2\alpha_4)}}$$

(could solve $x - \frac{\alpha_5}{\alpha_2} = \frac{2}{\alpha_2}\sqrt{x\frac{\alpha_5}{\alpha_4}(\alpha_3\alpha_5 - \alpha_2\alpha_4)}$ to get $x = \frac{D_2}{D_1} > \dots$)

(10) $\begin{cases} u_t = \alpha u^2 v - \mu u + D_1 u_{xx} \\ v_t = \beta - \alpha u^2 v + D_2 v_{xx} \end{cases}$

Spatially uniform steady states $\begin{cases} \alpha \bar{u}^2 \bar{v} - \mu \bar{u}\ 0 \\ \beta - \alpha \bar{u}^2 \bar{v} = 0 \end{cases}$ $\Rightarrow \beta - \mu \bar{u} = 0 \Rightarrow \bar{u} = \frac{\beta}{\mu} \Rightarrow$

$\bar{v} = \frac{\beta}{\alpha \bar{u}^2} = \frac{\mu^2}{\alpha\beta}$

$\Rightarrow (\bar{u}, \bar{v}) = \left(\frac{\beta}{\mu}, \frac{\mu^2}{\alpha\beta}\right)$

$$A(u,v) = \begin{pmatrix} 2\alpha uv - \mu & \alpha u^2 \\ -2\alpha uv & -\alpha u^2 \end{pmatrix} \Rightarrow A(\bar{u},\bar{v}) = \begin{pmatrix} \mu & \frac{\alpha\beta^2}{\mu^2} \\ -2\mu & -\frac{\alpha\beta^2}{\mu^2} \end{pmatrix} = A$$

1. $\det A = -\frac{\alpha\beta^2}{\mu} + 2\frac{\alpha\beta^2}{\mu} = \frac{\alpha\beta^2}{\mu} > 0$, OK for stability

2. $\operatorname{Tr} A = \mu - \frac{\alpha\beta^2}{\mu^2} < 0$ if $\underline{\mu^3 < \alpha\beta^2}$ condition for $(\bar{u},\bar{v})$ to be stable when $D_1 = D_2 = 0$.

Diffusive instability

3. $a_{11} D_2 + a_{22} D_1 > 2\sqrt{D_1 D_2 \det A} \Leftrightarrow$

$\underline{\mu D_2 - \frac{\alpha\beta^2}{\mu^2} D_1 > 2\sqrt{D_1 D_2 \alpha\beta^2/\mu}}$ condition on the parameters α, β, μ and D_1, D_2 for diffusive instability.

Choose $\alpha = 2, \beta = \mu = 1 \Rightarrow (\bar{u},\bar{v}) = (1, \frac{1}{2})$ and $\operatorname{Tr} A = 1 - 2 = -1 < 0$

$\Rightarrow (\bar{u},\bar{v})$ stable when $D_1 = D_2 = 0$

With $D_1 = 1$, condition 3. becomes

$D_2 - 2 > 2\sqrt{D_2 \cdot 2}$ $\left(x - 2 = 2\sqrt{2}\sqrt{x} \Rightarrow x^2 - 4x + 4 = 8x \Rightarrow x^2 - 12x + 4 = 0 \Rightarrow x = 6 \pm \sqrt{32} = 6 \underbrace{\pm}_{x>2} 4\sqrt{2} \right)$

$\Rightarrow \underline{D_2 > 6 + 4\sqrt{2}} \approx 11.66$, condition on D_2 for diffusive instability

Take $D_2 = 12$ and $0 < x < 25$ $(L = 25)$ $\Rightarrow q = \frac{n\pi}{25}$

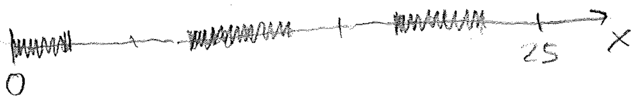
$u = \frac{a_{11} D_2 + a_{22} D_1}{D_1 D_2} = \frac{12 - 2}{12} = \frac{5}{6}$, $w = \frac{\det A}{D_1 D_2} = \frac{2}{12} = \frac{1}{6}$

$q_m^2 = \frac{u}{2} = \frac{5}{12}$, $\Delta = \sqrt{q_m^2 - w} = \sqrt{\frac{25}{144} - \frac{24}{144}} = \frac{1}{12}$ $\Rightarrow$

$\frac{5}{12} - \frac{1}{12} < q^2 < \frac{5}{12} + \frac{1}{12}$ gives unstable perturbations

$\Rightarrow \frac{1}{3} < \frac{n^2\pi^2}{25^2} < \frac{1}{2} \Rightarrow 21.1 < n^2 < 31.7 \rightarrow n = 5$ only solution

//// = intervals where $\cos\frac{5\pi x}{25} = \cos\frac{\pi x}{5} > 0$



(10) cont.

With the same constants, $\alpha = 2$, $\beta = \mu = D_1 = 1$, $D_2 = 12$, one gets in 2D

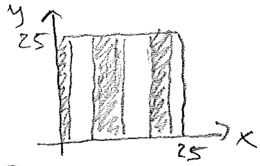
$$\frac{1}{3} < Q^2 < \frac{1}{2}, \quad Q^2 = q_1^2 + q_2^2$$

$$0 < x < 25,\ 0 < y < 25 \Rightarrow q_1 = \frac{m\pi}{25},\ q_2 = \frac{n\pi}{25}, \quad m, n = 0, 1, 2, \ldots$$

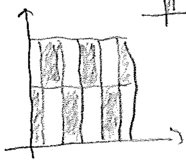
$$\Rightarrow \underline{21.1 < m^2 + n^2 < 31.7}$$

Possible combinations, and their patterns, are

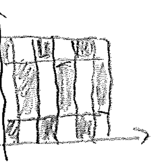
$(m,n) = (5,0)$, $m^2+n^2 = 25$, $\cos\frac{5\pi x}{25}$



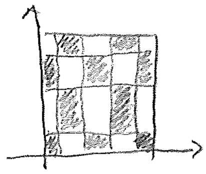
$(5,1)$, 26 $\cos\frac{5\pi x}{25}\cos\frac{\pi y}{25}$



$(5,2)$, 29 $\cos\frac{5\pi x}{25}\cos\frac{2\pi y}{25}$



$(4,3)$, 25 $\cos\frac{4\pi x}{25}\cos\frac{3\pi y}{25}$



$$\left.\begin{matrix}(3,4)\\(2,5)\\(1,5)\\(0,5)\end{matrix}\right\}$$ pictures like above, but with x and y swapped