

Fourier, sin-, and cos-series

Consider real one-variable functions on an interval $(-L, L)$, and define

$$(f, g) = \int_{-L}^L f(x)g(x) dx \quad (\text{other notations: } (f|g) \text{ or } \langle f|g \rangle)$$

Then $(f, g) = (g, f)$, $(f, c_1g_1 + c_2g_2) = c_1(f, g_1) + c_2(f, g_2)$, and $(f, f) \geq 0$ (and, if f is continuous, $= 0$ only if $f(x) = 0$ for all x). This means that (f, g) is a scalar product or an inner product (as defined in the linear algebra course).

Let $m \geq 0$ and $n \geq 0$ be integers and recall that $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$. Then

$$\begin{aligned} (\cos \frac{m\pi x}{L}, \cos \frac{n\pi x}{L}) &= \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \frac{1}{2} \int_{-L}^L (\cos \frac{(m+n)\pi x}{L} + \cos \frac{(m-n)\pi x}{L}) dx = \\ &= \left[\frac{L}{(m+n)\pi} \sin \frac{(m+n)\pi x}{L} + \frac{L}{(m-n)\pi} \sin \frac{(m-n)\pi x}{L} \right]_{-L}^L = 0 \quad \text{if } m \neq n. \end{aligned}$$

$$\text{If } m = n \neq 0, \quad (\cos \frac{m\pi x}{L}, \cos \frac{m\pi x}{L}) = \frac{1}{2} \int_{-L}^L (\cos \frac{2m\pi x}{L} + 1) dx = L,$$

$$\text{and if } m = n = 0, \quad \cos \frac{0\pi x}{L} = 1 \text{ and } (1, 1) = \int_{-L}^L dx = 2L.$$

$$\text{Similarly, for } m \geq 0 \text{ and } n \geq 1, \quad (\cos \frac{m\pi x}{L}, \sin \frac{n\pi x}{L}) = 0,$$

$$\text{and, for } m \geq 1 \text{ and } n \geq 1, \quad (\sin \frac{m\pi x}{L}, \sin \frac{n\pi x}{L}) = 0 \text{ if } m \neq n \text{ and } = L \text{ if } m = n.$$

$\Rightarrow 1, \cos \frac{\pi x}{L}, \sin \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \sin \frac{2\pi x}{L}, \dots$ are all orthogonal. In fact, they form an orthogonal basis for the (infinite-dimensional) vector space of all "nice" functions. Any (nice) $f(x)$ can be written as an infinite linear combination

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \quad \text{on } (-L, L), \text{ called the Fourier series of } f.$$

To find the coefficients a_n for a given f , take the scalar product with $\cos \frac{n\pi x}{L}$:

$$\begin{aligned} (f(x), \cos \frac{n\pi x}{L}) &= (\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L}), \cos \frac{n\pi x}{L}) = \\ &= \frac{a_0}{2} \underbrace{(1, \cos \frac{n\pi x}{L})}_{=0, n \geq 1, =2L, n=0} + \sum_{m=1}^{\infty} (\underbrace{a_m (\cos \frac{m\pi x}{L}, \cos \frac{n\pi x}{L})}_{=0, m \neq n, =L, m=n} + \underbrace{b_m (\sin \frac{m\pi x}{L}, \cos \frac{n\pi x}{L})}_{=0}) = a_n L \end{aligned}$$

$$\Rightarrow a_n = \frac{1}{L} (f(x), \cos \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad \text{for } n \geq 0.$$

$$\text{In the same way, } b_n = \frac{1}{L} (f(x), \sin \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad \text{for } n \geq 1.$$

Example. Let $L = \pi$ and $f(x) = x$. Then partial integration gives

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \left[-\frac{x \cos nx}{n\pi} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n\pi} dx = -\frac{2\pi \cos n\pi}{n\pi} + \left[\frac{\sin nx}{n^2\pi} \right]_{-\pi}^{\pi} = -\frac{2(-1)^n}{n} + 0$$

Similarly $a_n = 0$ for all $n \geq 0$ so

$$f(x) = x = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx = 2(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots) \quad \text{on } (-\pi, \pi)$$

More theory (convergence of series, integrals, function spaces, ...) in courses on Transform Theory or Fourier Analysis. One can also use $e^{i\pi nx/L}$ and write a complex version of the Fourier series.

Suppose now that $f(x)$ is defined on $(0, L)$ only. Extend f to an *even* function \tilde{f} on $(-L, L)$. This means that $\tilde{f}(x) = f(x)$ on $(0, L)$ and that $\tilde{f}(-x) = \tilde{f}(x)$. The Fourier series of \tilde{f} on $(-L, L)$ has all $b_n = 0$ since $\tilde{f}(x)$ even and $\sin \frac{n\pi x}{L}$ odd $\Rightarrow \tilde{f}(x) \sin \frac{n\pi x}{L}$ odd, and

$$b_n = \frac{1}{L} \int_{-L}^L \tilde{f}(x) \sin \frac{n\pi x}{L} dx = 0 \quad (\text{since the interval is symmetric})$$

Then $\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$ on $(-L, L)$, with

$$a_n = \frac{1}{L} \int_{-L}^L \underbrace{\tilde{f}(x)}_{\text{even}} \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L \underbrace{\tilde{f}(x)}_{=f(x)} \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Since $f = \tilde{f}$ on $(0, L)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad \text{on } (0, L), \text{ with } a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

This is called the cos-series of f on $(0, L)$.

In the same way (extend f to an odd function),

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad \text{on } (0, L), \text{ with } b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx ,$$

the sin-series of f on $(0, L)$.

TEST QUESTIONS

1. Find the Fourier series on $(-L, L)$ of $f(x) = \begin{cases} 1 & \text{om } x \geq 0 \\ 0 & \text{om } x < 0 \end{cases}$
2. Find the Fourier series of $f(x) = 8 \sin 3x - 17 \cos 7x$ on $(-\pi, \pi)$
3. Find the cos- and sin-series of $f(x) = x$ on $(0, 1)$

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SOLUTIONS

$$1. \Rightarrow a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{L} \int_0^L \cos \frac{n\pi x}{L} dx = \frac{1}{L} \left[\frac{L \sin \frac{n\pi x}{L}}{n\pi} \right]_0^L = 0 \quad \text{for } n \neq 0,$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{L} \int_0^L dx = 1 \quad ,$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{L} \int_0^L \sin \frac{n\pi x}{L} dx = \frac{1}{L} \left[\frac{-L \cos \frac{n\pi x}{L}}{n\pi} \right]_0^L = \frac{1 - \cos n\pi}{n\pi} \quad \text{for } n \geq 1,$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin \frac{n\pi x}{L} = \frac{1}{2} + \frac{2}{\pi} \left(\sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \dots \right) \quad \text{on } (-L, L)$$

2. The Fourier series is unique so $f(x) = 8 \sin 3x - 17 \cos 7x$ has Fourier series $8 \sin 3x - 17 \cos 7x$

$$3. a_n = 2 \int_0^1 x \cos(n\pi x) dx = 2 \left[\frac{x \sin(n\pi x)}{n\pi} \right]_0^1 - 2 \int_0^1 \frac{\sin(n\pi x)}{n\pi} dx = 0 + 2 \left[\frac{\cos(n\pi x)}{n^2 \pi^2} \right]_0^1 = 2 \frac{\cos(n\pi) - 1}{n^2 \pi^2}$$

for $n \neq 0$, and $a_0 = 2 \int_0^1 x dx = [x^2]_0^1 = 1 \Rightarrow$ the cos-series is

$$x = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(n\pi x) = \frac{1}{2} - \frac{4}{\pi^2} \cos(\pi x) - \frac{4}{9\pi^2} \cos(3\pi x) - \frac{4}{25\pi^2} \cos(5\pi x) - \dots \quad \text{on } (0, 1)$$

$$b_n = 2 \int_0^1 x \sin(n\pi x) dx = -2 \left[\frac{x \cos(n\pi x)}{n\pi} \right]_0^1 + 2 \int_0^1 \frac{\cos(n\pi x)}{n\pi} dx = -2 \frac{\cos(n\pi)}{n\pi} + 2 \left[\frac{\sin(n\pi x)}{n^2 \pi^2} \right]_0^1 = -2 \frac{(-1)^n}{n\pi}$$

\Rightarrow the sin-series is

$$x = \frac{-2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) = \frac{2}{\pi} (\sin(\pi x) - \frac{1}{2} \sin(2\pi x) + \frac{1}{3} \sin(3\pi x) - \dots) \quad \text{on } (0, 1)$$