Linear ODE's of order 1 and 2

$f(t)$, $g(t)$, $h(t)$ given (one-variable) functions. Find the function $x(t)$ such that

order 1: $x'(t) + f(t)x(t) = g(t)$ (1)

order 2: $x''(t) + f(t)x'(t) + g(t)x(t) = h(t)$ (2)

Both have solution structure $x(t) = x_h(t) + x_p(t)$, where $x_h(t)$ are all homogeneous solutions (the solutions if $g(t) = 0$ in (1) and $h(t) = 0$ in (2)), $x_p(t)$ is one particular solution [true for all linear problems, also systems of linear equations, linear difference equations etc.]

(1) can be solved by multiplying by the integrating factor $e^{F(t)}$, where $F' = f$:

$$\underbrace{e^{F(t)}x'(t) + f(t)e^{F(t)}x(t)}_{=\frac{d}{dt}(e^{F(t)}x(t))} = g(t)e^{F(t)} \quad \Rightarrow \quad e^{F(t)}x(t) = \int g(t)e^{F(t)}dt + C \quad \Rightarrow$$

$$\Rightarrow \quad x(t) = \underbrace{e^{-F(t)}\int g(t)e^{F(t)}dt}_{x_p(t)\ \ (=0 \text{ if } g(t)=0)} + \underbrace{Ce^{-F(t)}}_{x_h(t)}$$

(2) is more difficult for general f, g and h. If $f(t) = a$ and $g(t) = b$ constants, then $x''(t) + ax'(t) + bx(t) = h(t)$ has homogeneous solutions $x_h(t) = c_1e^{r_1t} + c_2e^{r_2t}$ if $r_1 \neq r_2$ ($x_h(t) = (c_1t + c_2)e^{r_1t}$ if $r_1 = r_2$), where $r_{1,2}$ are solutions to $r^2 + ar + b = 0$. If $r_{1,2} = \alpha \pm i\beta$ are complex, this can be written $x_h(t) = e^{\alpha t}(c_3 \cos\beta t + c_4 \sin\beta t)$. $x_p(t)$ is found by some Ansatz (qualified guess).

Separable ODE's of order 1 (can be linear or non-linear)

$f(x)$, $g(t)$ given (one-variable) functions. Find the function $x(t)$ such that

$f(x)x'(t) = g(t)$

Primitive function on $t \quad \Rightarrow \quad F(x(t)) = G(t) + C$
This gives a relation between x and t from which $x(t)$ sometimes can be extracted explicitly.
The solution structure is (in general) not homog. + particular (if non-linear).

Ex. 1D autonomous dynamical system: $\dfrac{dx}{dt} = h(x)$ (no explicit t in the equation) is a separable equation $\dfrac{1}{h(x)} \cdot \dfrac{dx}{dt} = 1 \quad (f(x) = \frac{1}{h(x)},\ g(t) = 1)$

TEST QUESTIONS

1. Solve $x'(t) + 3x(t) = 6$
2. Solve $x'(t) + tx(t) = t$
3. Solve $x''(t) + 6x'(t) + 5x(t) = 0$
4. Solve $x''(t) + 6x'(t) + 9x(t) = 0$
5. Solve $x''(t) + 2x'(t) + 5x(t) = 0$
6. Solve $x(t)^2x'(t) = 2t^5$
7. Solve $x'(t) = x(t)(1 - x(t))$ Hint: $\frac{1}{a(1-a)} = \frac{1}{a} + \frac{1}{1-a}$ (note: one often writes just $x' = x(1 - x)$)

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ANSWERS

1. $x(t) = 2 + Ce^{-3t}$ (integrating factor e^{3t})
2. $x(t) = 1 + Ce^{-t^2/2}$ (integrating factor $e^{t^2/2}$)
3. $x(t) = C_1e^{-t} + C_2e^{-5t}$
4. $x(t) = (C_1t + C_2)e^{-3t}$
5. $x(t) = C_1e^{(-1+2i)t} + C_2e^{(-1-2i)t} = e^{-t}(C_3 \cos 2t + C_4 \sin 2t)$
6. $x(t) = (t^6 + 3C)^{1/3} = (t^6 + K)^{1/3}$ (separable)
7. $(\frac{1}{x} + \frac{1}{1-x})x' = 1 \;\Rightarrow\; \ln|x| - \ln|1-x| = t + C \Rightarrow\; \ln|\frac{x}{1-x}| = t + C \;\Rightarrow\; |\frac{x}{1-x}| = e^{t+C} = e^te^C \;\Rightarrow$ $\frac{x}{1-x} = \pm e^Ce^t = Ke^t \;\Rightarrow\; xe^{-t} = (1-x)K \;\Rightarrow\; x(K + e^{-t}) = K \;\Rightarrow\; x(t) = \frac{K}{K+e^{-t}}$