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Some linear algebra

A $n \times n$ matrix. $A\bar{v} = \lambda\bar{v}$, $\bar{v} \neq \bar{0}$, λ scalar $\Rightarrow$ $\bar{v}$ eigenvector of A with eigenvalue λ

Step 1. Find all λ by solving $\det(A - \lambda I) = 0$, I = identity matrix. $\det(A - \lambda I)$ is a polynomial of degree n in λ.

Step 2. For each λ, find the corresponding $\bar{v}$ by solving the linear system $(A - \lambda I)\bar{v} = \bar{0}$

$\mathrm{Tr}A$ = trace of A = sum of diagonal elements $= a_{11} + a_{22} + \cdots + a_{nn}$

$$n = 2 \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} =$$

$$\lambda^2 - \underbrace{(a_{11} + a_{22})}_{=\mathrm{Tr}A}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{=\det A} = \lambda^2 - (\mathrm{Tr}A)\lambda + \det A$$

Also, factorizating, $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$

$\Rightarrow \begin{cases} \mathrm{Tr}A = \lambda_1 + \lambda_2 \\ \det A = \lambda_1\lambda_2 \end{cases}$ always. The same is true for $n \times n$ matrices : $\begin{cases} \mathrm{Tr}A = \lambda_1 + \ldots + \lambda_n \\ \det A = \lambda_1 \cdot \ldots \cdot \lambda_n \end{cases}$

(also if some λ_j:s are complex)

$$\det(A - \lambda I) = 0 \Rightarrow \lambda_{1,2} = \frac{\mathrm{Tr}A}{2} \pm \frac{\sqrt{(\mathrm{Tr}A)^2 - 4\det A}}{2}$$

$(\mathrm{Tr}A)^2 - 4\det A = \mathrm{disc}(A)$ = discriminant of A. Sign of $\mathrm{disc}(A)$ determines if $\lambda_{1,2}$ real or complex

Observations

$\lambda_1 > 0,\ \lambda_2 > 0 \Rightarrow \mathrm{Tr}A > 0,\ \det A > 0$
$\lambda_1 < 0,\ \lambda_2 < 0 \Rightarrow \mathrm{Tr}A < 0,\ \det A > 0$
$\lambda_1 > 0,\ \lambda_2 < 0 \Rightarrow \det A < 0$
$\lambda_{1,2} = a \pm ib$ (complex) $\Rightarrow \mathrm{Tr}A = 2a,\ \det A = a^2 + b^2 > 0$

Important for systems of ODE's (of course, some λ_j may be 0)

TEST QUESTIONS (for a real 2×2 matrix A)

1. If $\mathrm{Tr}A = 5$ and $\det A = 4$, find λ_1 and λ_2. Is it OK with the observations?
2. If $\det A = -3$, why are λ_1 and λ_2 real. What are their signs?
3. If $\det A = -3$, can one decide the sign of $\mathrm{Tr}A$?
4. If $\mathrm{Tr}A = -5$ and $\det A = 3$, without calculating λ_1 and λ_2, what are the signs of their real parts?
5. If $\det A = 0$ and $\mathrm{Tr}A = 3$, find λ_1 and λ_2
6. If $\mathrm{Tr}A = 0$ and $\det A = -4$, find λ_1 and λ_2
7. If $\mathrm{Tr}A = 0$ and $\det A = 4$, find λ_1 and λ_2

8*. Find an $A \neq 0$ with $\mathrm{Tr}A = \det A = 0$.

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ANSWERS

1. $\lambda_1 = 4$, $\lambda_2 = 1$ (consistent with $\text{Tr}A > 0$ and $\det A > 0$)
2. $\det A = \lambda_1 \lambda_2 < 0$ can only hold if $\lambda_1 > 0$ and $\lambda_2 < 0$ real
3. No, $\text{Tr}A = \lambda_1 + \lambda_2$ can be both positive and negative if $\lambda_1 \lambda_2 < 0$
4. $\text{Tr}A < 0$ and $\det A > 0$ gives either $\lambda_{1,2} < 0$ real or $\lambda_{1,2} = a \pm ib$ with $a < 0$, in both cases $\text{Re}(\lambda_{1,2}) < 0$
5. $\lambda_1 = 3$, $\lambda_2 = 0$
6. $\lambda_1 = 2$, $\lambda_2 = -2$
7. $\lambda_{1,2} = \pm 2i$
8. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$