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Linear difference equations of order 1 and 2

$f(n)$, $g(n)$, $h(n)$ given functions of n, where $n \geq 0$ is an integer. Find the function (sequence) $x(n) = x_n$ such that

order 1: $x_{n+1} + f(n)x_n = g(n)$ (1)

order 2: $x_{n+2} + f(n)x_{n+1} + g(n)x_n = h(n)$ (2)

Both have solution structure $x_n = x_{n,h} + x_{n,p}$, where $x_{n,h}$ are all homogeneous solutions (the solutions if $g(n) = 0$ in (1) and $h(n) = 0$ in (2)), $x_{n,p}$ is one particular solution.

These are more difficult to solve than the corresponding differential equations because there are no primitive functions. For constant coefficients the homogeneous solution is easy to find:

$x_{n+1} + ax_n = 0$ has the solution $x_n = c(-a)^n$ (3).

$x_{n+2} + ax_{n+1} + bx_n = 0$ has the solution $x_n = c_1(r_1)^n + c_2(r_2)^n$ (4) ,

where $r_1 \neq r_2$ are the solutions to $r^2 + ar + b = 0$. In case $r_1 = r_2$, then $x_n = (c_1 n + c_2)r_1^n$. If $r_{1,2} = \alpha \pm i\beta = \rho e^{\pm i\varphi}$ (polar form) are complex, one can write $r_{1,2}^n = \rho^n e^{\pm in\varphi}$.

The constant c in (3) can be determined by an initial value of x_0. In (4) c_1 and c_2 are determined by giving x_0 and x_1.

For non-homogeneous equations, a particular solution is found by some Ansatz.

Observe that the condition for $x_n \to 0$ as $n \to \infty$ in (3) is $|a| < 1$, and in (4) $|r_{1,2}| < 1$.

Example

Solve $x_{n+2} - x_{n+1} - 6x_n = 0$ with initial conditions $x_0 = 3$ and $x_1 = 4$

Solution: the solutions to $r^2 - r - 6 = 0$ are $r_1 = 3$ and $r_2 = -2$ $\Rightarrow$ the general solution is $x_n = c_1 3^n + c_2(-2)^n$

The initial condition at $n = 0$ gives $x_0 = c_1 3^0 + c_2(-2)^0 = c_1 + c_2 = 3$ (1), and at $n = 1$ we get $x_1 = c_1 3^1 + c_2(-2)^1 = 3c_1 - 2c_2 = 4$ (2). (1) and (2) $\Rightarrow$ $c_1 = 2$ and $c_2 = 1$ $\Rightarrow$ the solution with the given initial conditions is $x_n = 2 \cdot 3^n + (-2)^n$

TEST QUESTIONS

1. Solve $x_{n+1} + 3x_n = 0$ with initial condition $x_0 = 5$
2. Solve $x_{n+2} + 6x_{n+1} + 5x_n = 0$ with initial conditions $x_0 = 5$ and $x_1 = 3$
3. Solve $x_{n+2} + 6x_{n+1} + 9x_n = 0$ with initial conditions $x_0 = 5$ and $x_1 = 3$
4. Solve $x_{n+2} - 2x_{n+1} + 2x_n = 0$ with initial conditions $x_0 = 5$ and $x_1 = 3$

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ANSWERS

1. General solution is $x_n = c(-3)^n$.
Initial condition $x_0 = 5 \Rightarrow c = 5 \Rightarrow x_n = 5(-3)^n$.

2. General solution is $x_n = c_1(-1)^n + c_2(-5)^n$.
Initial conditions $x_0 = 5,\ x_1 = 3 \ \Rightarrow\ c_1 + c_2 = 5,\ -c_1 - 5c_2 = 3 \Rightarrow c_1 = 7,\ c_2 = -2 \Rightarrow$
$x_n = 7(-1)^n - 2(-5)^n$

3. General solution is $x_n = (c_1 n + c_2)(-3)^n$
Initial conditions $x_0 = 5,\ x_1 = 3 \Rightarrow c_2 = 5,\ (c_1 + c_2)(-3) = 3 \Rightarrow c_1 = -6,\ c_2 = 5 \Rightarrow$
$x_n = (-6n + 5)(-3)^n$

4. General solution is $x_n = c_1(1+i)^n + c_2(1-i)^n = (\sqrt{2})^n(c_1 e^{in\pi/4} + c_2 e^{-in\pi/4}) =$
$= 2^{n/2}(c_3 \cos\frac{n\pi}{4} + c_4 \sin\frac{n\pi}{4})$
Initial conditions $x_0 = 5,\ x_1 = 3 \ \Rightarrow\ c_3 = 5,\ 2^{1/2}(c_3\frac{1}{\sqrt{2}} + c_4\frac{1}{\sqrt{2}}) = 3 \Rightarrow c_3 = 5,\ c_4 = -2 \Rightarrow$
$x_n = 2^{n/2}(5\cos\frac{n\pi}{4} - 2\sin\frac{n\pi}{4})$