Modeling with (systems of) differential equations (ODE's)

$N(t)$ = size of population at time t
should be an integer, but if population large ($>10^2$?) we model with real values.

<u>First model</u>: the change in $N(t)$ is proportional to its size

$$\underbrace{\frac{dN(t)}{dt} = \underset{\substack{\uparrow \\ \text{constant}}}{r} \cdot N(t)}_{\text{linear ODE, order 1}} \quad , \; r>0 \text{ if increasing population}$$

linear ODE, order 1, the solution is $N(t) = \underset{\substack{\uparrow \\ \text{arbitrary constant}}}{k} e^{rt}$

If at time $t=0$ the population is N_0: $N(0)=N_0$,
then $N_0 = k \cdot e^{r \cdot 0} = k \Rightarrow \underline{N(t) = N_0 e^{rt}}$ <u>exponential growth</u>

One interpretation of r: When is $N(t) = 2N_0$?
$N_0 e^{rt} = 2N_0 \Rightarrow rt = \ln 2 \Rightarrow t = \frac{\ln 2}{r}$ = time for population doubling

large $r \Rightarrow$ short doubling time

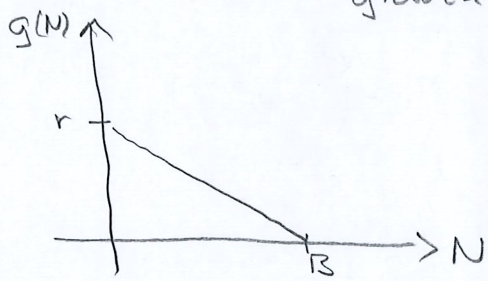
The same ODE with $r<0$ models radioactive decay. Then $\frac{\ln 2}{|r|}$ is the half-life of the radioactive material.

<u>Second model</u> Logi<u>stic</u> <u>growth</u> [exponential growth cannot go on forever]

Suppose $\frac{dN}{dt} = g(N)N$ with $g(N)$ not constant

Let B be the <u>carrying capacity</u>; if $N(t) = B$ there is no more growth

A simple model is $g(N) = r\left(1 - \frac{N}{B}\right)$



For N much smaller than B,
$g(N) \approx r \Rightarrow$ exponential growth

For N near B, $g(N)$ is very small
$\Rightarrow$ slow growth

The model $\frac{dN(t)}{dt} = r\left(1 - \frac{N(t)}{B}\right)N(t)$ is called the logi<u>stic</u> <u>equation</u>.

It is a non-linear ODE (N^2-term) of order 1. Next seminar we see how to solve it and understand solutions.

[it is separable]

Systems of ODE's

2D: Find functions $x(t)$ and $y(t)$ such that

$$\begin{cases} \frac{dx}{dt} = F(x,y,t) \\ \frac{dy}{dt} = G(x,y,t) \end{cases}$$

F, G some given expressions

The system is called autonomous if F and G do not explicitly depend on time:

$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

Since time scale and origin can be chosen by the user (us!), most real-world problems are autonomous [they are the ones we focus on, the laws of nature do not change in time]

Example 1: Predator-prey model

$x(t)$ = prey population at time t (hare, rabbit, ...)
$y(t)$ = predator — " — (wolf, lynx, ...)

First model: 1. x grows exp. if no predators ($y=0$) [could be modified to logistic]

2. y decays exp. if no prey ($x=0$) [maybe exp. not best model, may die out in finite time]

3. growth of y proportional to likelihood of finding prey, modeled by term xy, higher x and y increases probability [one factor fixed $\Rightarrow$ linear in the other]. x should decrease by a similar term

Leads to simplest Lotka-Volterra model

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + dxy \end{cases}$$

$a,b,c,d > 0$ constants

Non-linear because of xy-term. To be analyzed in detail, also extensions of it.

Ex 2: SIR-models

$S(t)$ = number of susceptible for an infectious disease
$I(t)$ = number of infected
$R(t)$ = number of removed (immune or deceased)

If the number of encounters between infected and susceptible is modeled by an $S \cdot I$-term (random encounters) one gets

$$\begin{cases} \frac{dS}{dt} = -\beta IS \\ \frac{dI}{dt} = \beta IS - \nu I \end{cases}$$

(and $\frac{dR}{dt} = \nu I$)

νI: exp. decay in I if $S=0$

Basic model in studies in spread of, e.g., covid-19. We will see several generalizations of this model.

Ex 3: Chemostat

$$\begin{cases} \dfrac{dN}{dt} = \alpha_1 \dfrac{C}{1+C} N - N \\ \dfrac{dC}{dt} = -\underbrace{\dfrac{C}{1+C} N}_{\text{non-linear term}} - C + \alpha_2 \end{cases}$$

$\uparrow$ also non-lin but no problem to handle

$N(t)$ = bacterial density
$C(t)$ = nutrient concentration
α_1, α_2 constants

Interpretation and analysis, seminars 4-6

Note: Ex 1-3 are autonomous

A linear auton. system can be written

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) = \alpha x + \beta y \\ \dfrac{dy}{dt} = G(x,y) = \gamma x + \delta y \end{cases} \iff \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\text{constant matrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

[corresp. in $\geq 3D$]

General questions

* How to solve — single ODE's / systems of ODE's — linear, non-linear

* If we cannot solve explicitly (normal situation in non-linear case), how can we still understand solutions qualitatively? Explicit solutions is not the only way to get information.

* Interpretation of solutions (biological, chemical, physical, ...). Agreement with experiments/observations/simulations...

* Understand applicability and limitations of models (e.g. SIR type models). In biology the equations are models, in physics they can be fundamental laws of nature.

* Easy to do simulations/numerical solutions, but theory needed to understand and interpret.

Have a look at repetition of ODE's with test questions from notes of lecture 2 (and maybe repetition of linear algebra from lecture 3).