SIS, SIR, SIRS and other epidemic models

S = susceptibles, I = infective, R = removed

$S + I + R = N$ = constant = total population

The SIS model (still no class R)

$$\begin{tikzcd} \boxed{S} \arrow[r] & \boxed{I} \arrow[l, bend right] \end{tikzcd}$$

$$\begin{cases} \dfrac{dS}{dt} = \nu I - \beta S I \\[2mm] \dfrac{dI}{dt} = \beta S I - \nu I \end{cases}$$

$\beta S I$ models number of new infected

νI number of recovered, again susceptible

$\frac{1}{\nu}$ is expected time in state I

$S + I = N \Rightarrow S = N - I \Rightarrow$

$$\frac{dI}{dt} = \beta(N-I)I - \nu I = (\beta N - \nu) I \left(1 - \frac{\beta I}{\beta N - \nu}\right)$$, a logistic eq. for $I(t)$

$$\Rightarrow I(t) = \frac{B I(0)}{I(0) + (B - I(0)) e^{-rt}}$$, where $r = \beta N - \nu$ and $B = \frac{\beta N - \nu}{\beta} = N\left(1 - \frac{\nu}{\beta N}\right)$

As $t \to \infty$, $I(t) \to \begin{cases} B & \text{if } r > 0 \Rightarrow \text{disease established in population} \\ 0 & \text{if } r < 0 \Rightarrow \text{disease disappears (everyone in } S: S(t) \to N) \end{cases}$

$r < 0 \Leftrightarrow \beta N < \nu \Leftrightarrow \frac{\beta N}{\nu} < 1$.

Def $R_0 = \frac{\beta N}{\nu}$, basic reproduction ratio. $R_0 > 1 \Rightarrow I(t) \to N\left(1 - \frac{1}{R_0}\right)$

If $I(0)$ small and $S(0) \approx N$, $\frac{dI}{dt} = I(\beta S - \nu) \approx I(\beta N - \nu)$ for t small

$\Rightarrow I(t) \sim e^{(\beta N - \nu)t}$ for t small

$\left.\begin{array}{l} \Rightarrow \beta N - \nu \text{ can be observed early in the epidemic} \\ \frac{1}{\nu} \text{ can be observed from time in state } I \end{array}\right\} \Rightarrow$

$$R_0 = \frac{\beta N}{\nu} = \frac{\beta N - \nu + \nu}{\nu} = \frac{\beta N - \nu}{\nu} + 1$$ can be observed

The SIR model

Individuals can now go from I to R. Expected time in I is $\frac{1}{\nu}$

$$\begin{cases} \frac{dS}{dt} = -\beta I S \\ \frac{dI}{dt} = \beta I S - \nu I \\ \frac{dR}{dt} = \nu I \end{cases}$$

3D but $R = N - I - S$ ⇒ can study 2D system for S and I (then $R = N - I - S$):

$\boxed{S} \to \boxed{I} \to \boxed{R}$ (no reinfections)

$$\begin{cases} \frac{dS}{dt} = -\beta I S & = F_1(S,I) \\ \frac{dI}{dt} = \beta I S - \nu I & = F_2(S,I) \end{cases}$$

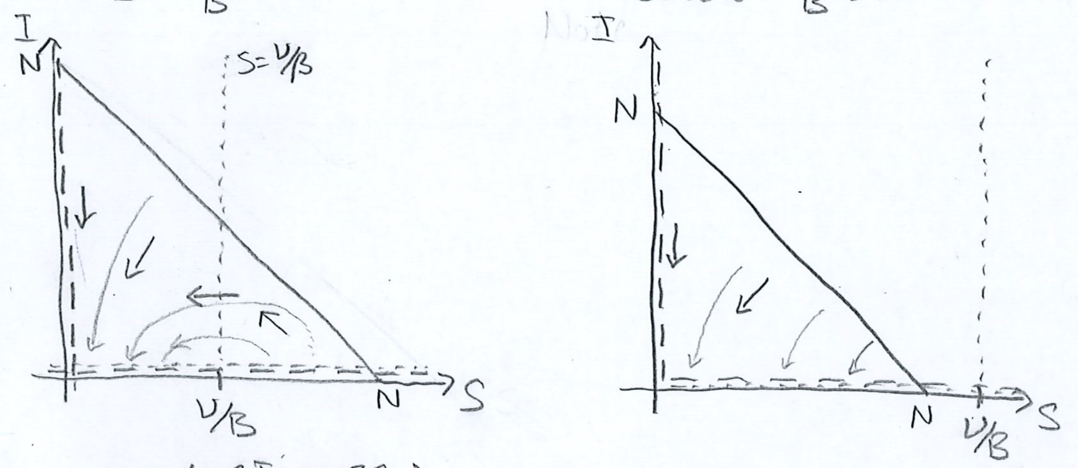
Do a phase plane analysis

S nullclines $-\beta I S = 0$ ⇒ $I = 0$ or $S = 0$ (coord. axes)

I nullclines $\beta I S - \nu I = I(\beta S - \nu) = 0$ ⇒ $\underbrace{I = 0}_{\text{axes}}$ or $\underbrace{S = \nu/\beta}_{\text{line}}$

Note that $I + S \leq N$ (entire population) ⇒ we must be in a triangle in the phase plane

Case 1: $\frac{\nu}{\beta} < N$ Case 2: $\frac{\nu}{\beta} > N$



Note: all points on S-axis ($I=0$) are steady states $\bar{F} = (F_1, F_2) = (0,0)$ there

Draw $\bar{F}$ as usual.

$$J(S,I) = \begin{pmatrix} -\beta I & -\beta S \\ \beta I & \beta S - \nu \end{pmatrix} \Rightarrow J(S,0) = \begin{pmatrix} 0 & -\beta S \\ 0 & \beta S - \nu \end{pmatrix} \quad \underbrace{\lambda_1 = 0}_{\text{sensitive case}}, \ \lambda_2 = \beta S - \nu$$

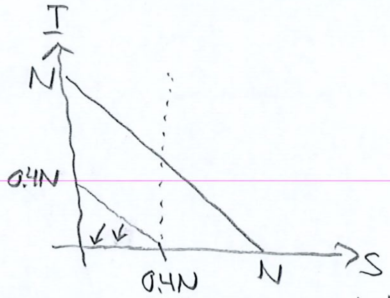
In case 2, $\lambda_2 < 0$ for all $S < N$. A small change in initial values will lead to a small change in $S_\infty = \lim_{t\to\infty} S(t)$. ~ stable

In case 1, $\lambda_2 < 0$ for all $S < \frac{\nu}{\beta}$ ⇒ similar to case 2. $\lambda_2 > 0$ for $\frac{\nu}{\beta} < S < N$ ⇒ unstable, solutions with $I(0) > 0$ small and $S(0) > \frac{\nu}{\beta}$ will move away with $I(t)$ increasing and $S(t)$ decreasing, when $S(t)$ becomes less than $\frac{\nu}{\beta}$, $I(t)$ will decrease and $\to 0$. A larger $S(0)$ gives a smaller S_∞.

See Maple plots for cases 1 and 2. (page 11.5-6)

As for SIS, we define $R_0 = \frac{\beta N}{\nu}$, and it can be estimated (observed) in the same way.

Case 1 is $R_0 > 1$, case 2 is $R_0 < 1$.

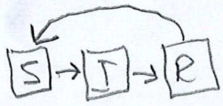


Ex: $R_0 = 2.5$, $\frac{\nu}{\beta} = \frac{N}{R_0} = 0.4N \Rightarrow$
if $R(t) > 0.6N$, then $S+I = N-R < 0.4N$ and we are in the small triangel. Then both $S'(t) < 0$ and $I'(t) < 0$, and $I(t) \to 0$.
The disease vanishes. Herd immunity!

This can also be achieved by vaccinating $60\% = 1 - \frac{1}{R_0}$ of population.

So in case I, if $S(0) \approx N$ and $I(0) > 0$ small, $I(t)$ will increase until $S(t) < \frac{N}{R_0}$ (or $R(t) > N(1 - \frac{1}{R_0})$), then $I(t)$ decreases. The total number of infected during the epidemy is $R_\infty = S(0) - S_\infty$, which can be estimated (numerically) for different initial conditions.

In case II, with $R_0 < 1$, $S'(t) < 0$ and $I'(t) < 0$ and the disease will disappear quickly.



SIRS model

With temporal immunity, one can move from R back to S

$$\begin{cases} \frac{dS}{dt} = \gamma R - \beta I S \\ \frac{dI}{dt} = \beta I S - \nu I \\ \frac{dR}{dt} = \nu I - \gamma R \end{cases}$$

$1/\gamma$ = expected time in R

Still, $R = N - S - I$ can be removed to get a 2D system for S and I;

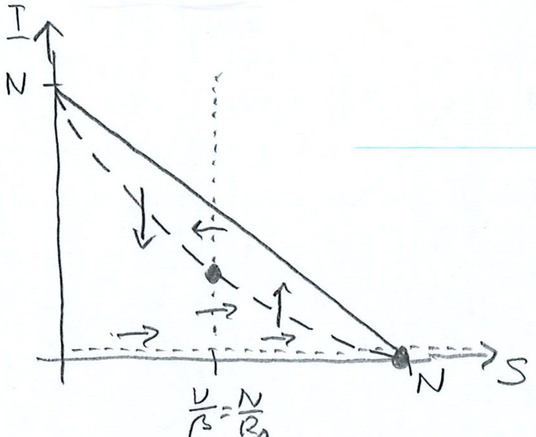
$$\begin{cases} \frac{dS}{dt} = \gamma(N - I - S) - \beta I S \\ \frac{dI}{dt} = \beta I S - \nu I \end{cases}$$

S nullclines $\gamma(N-I-S) - \beta I S = 0 \Rightarrow I = \gamma \frac{N-S}{\gamma + \beta S}$ curve through $(N,0)$ and $(0,N)$ with $\frac{dI}{dS} < 0$, $\frac{d^2 I}{dS^2} > 0$ (check!)

I nullclines $I(\beta S - \nu) = 0 \Rightarrow I = 0$ or $S = \nu/\beta$

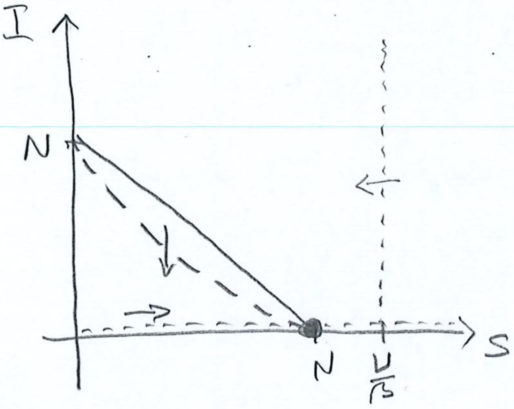
Put again $R_0 = \frac{\beta N}{\nu}$. We get two cases, $R_0 > 1$ and $R_0 < 1$

Case 1, $R_0 > 1$



Two steady states: $(N, 0)$ and $(\bar{S}, \bar{I}) = \left(\nu/\beta, \frac{\gamma}{\nu+\gamma}(N - \frac{\nu}{\beta})\right)$

Case 2, $R_0 < 1$



Steady state: $(N, 0)$.

Draw $\bar{F} = (F_1, F_2)$ as usual

$$J(S,I) = \begin{pmatrix} -\gamma-\beta I & -\gamma-\beta S \\ \beta I & \beta S - \nu \end{pmatrix} \Rightarrow J(N,0) = \begin{pmatrix} -\gamma & -\gamma-\beta N \\ 0 & \beta N - \nu \end{pmatrix} \quad \lambda_1 = -\gamma < 0, \quad \lambda_2 = \beta N - \nu \begin{cases} > 0 & \text{case I, saddle} \\ < 0 & \text{case II, stable} \end{cases}$$

$$J(\bar{S}, \bar{I}) = \begin{pmatrix} -\gamma-\beta\bar{I} & -\gamma-\nu \\ \beta\bar{I} & 0 \end{pmatrix} = J \qquad \left.\begin{array}{l} \operatorname{Tr} J = -\gamma - \beta\bar{I} < 0 \\ \det J = \beta\bar{I}(\gamma+\nu) > 0 \end{array}\right\} \Rightarrow (\bar{I}, \bar{S}) \text{ stable}$$

As $t \to \infty$, $(S, I, R) \to \begin{cases} (\nu/\beta, \bar{I}, N - \nu/\beta - \bar{I}) & \text{if } R_0 > 1 \text{ disease established in population} \\ (N, 0, 0) & \text{if } R_0 < 1 \text{ disease vanishes} \end{cases}$

Interpretation of $\frac{\nu}{\beta} > N$ ($R_0 < 1$): $\begin{cases} \text{large } \nu \Rightarrow \text{shorter time in } I \\ \text{small } \beta \Rightarrow \text{lower infection rate} \end{cases}$ (natural)

See Maple plots (page 11.7-8) [no meaning to $S(0) - S_\infty$ with reinfections]

<u>SEIR model</u> E = exposed (but not yet infective) new class between S and I (can also do SEIRS)

$$\begin{cases} S' = -\beta I S \\ E' = \beta I S - \delta E \\ I' = \delta E - \nu I \\ R' = \nu I \end{cases}$$

$1/\delta$ = expected time in E

$S + E + I + R = N \Rightarrow$ can reduce to 3D (not to 2D)

$\boxed{S} \to \boxed{E} \to \boxed{I} \to \boxed{R}$

steady states $\left.\begin{array}{l} -\beta I S = 0 \\ \beta I S - \delta E = 0 \\ \delta E - \nu I = 0 \end{array}\right\} \Rightarrow E = I = 0$, S any value

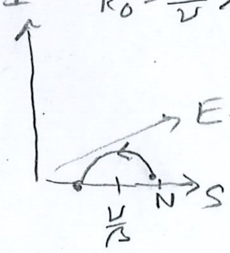
$$J = \begin{pmatrix} -\beta I & 0 & -\beta S \\ \beta I & -\delta & \beta S \\ 0 & \delta & -\nu \end{pmatrix} \underset{E=I=0}{=} \begin{pmatrix} 0 & 0 & -\beta S \\ 0 & -\delta & \beta S \\ 0 & \delta & -\nu \end{pmatrix}$$

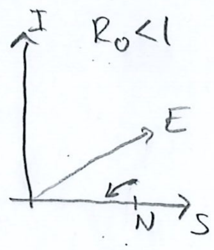
$\lambda_1 = 0$, $\lambda_{2,3}$ from $\tilde{J} = \begin{pmatrix} -\delta & \beta S \\ \delta & -\nu \end{pmatrix}$, $\operatorname{Tr}\tilde{J} < 0$, $\det\tilde{J} = \delta(\nu - \beta S) \begin{cases} > 0 & \text{if } S < \nu/\beta \text{ "stable"} \\ < 0 & \text{if } S > \nu/\beta \text{ unstable} \end{cases}$

Similar to SIR but one more dim. (E)

I $\quad R_0 = \frac{\beta N}{\nu} > 1$

(3D)





$R_0 < 1$

curves in 3D.

$R_0 > 1 \Rightarrow I(t)$ (and $E(t)$) can increase before $\to 0$

$R_0 < 1 \Rightarrow I(t)$ (and $E(t)$) decreases $\to 0$

[plots page 11.9]

SEIR models have been used for covid-19 ($\frac{1}{\delta} \sim 5$ days, $\frac{1}{\nu} \sim 5$ days ?)

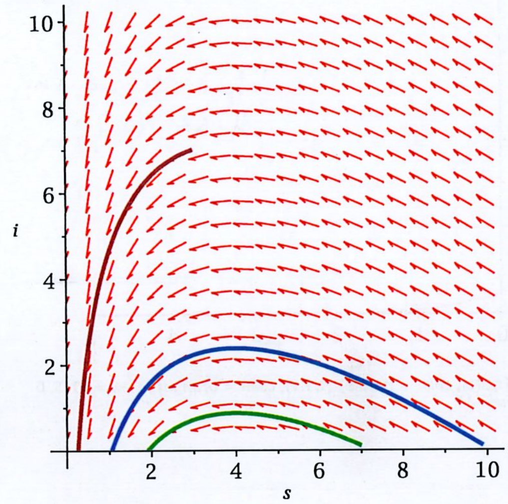
11.5

SIR model , $N=10$, $\beta=1$, $\nu=4$ $\Rightarrow$ $\frac{\nu}{\beta}<N$ (case 1) and $R_0=2.5>1$

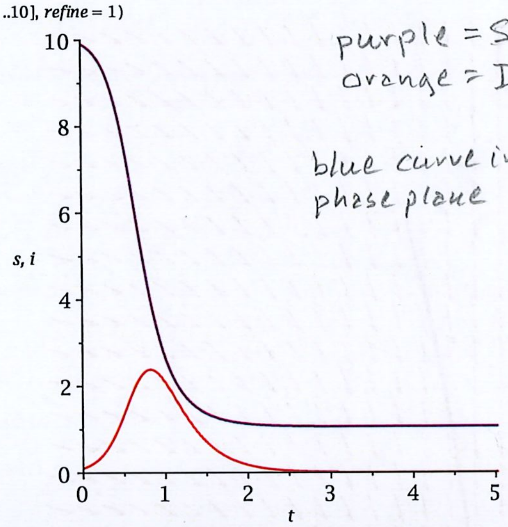
```
> with(DEtools) :
> sys := {diff(s(t), t) = -1· s(t)·i(t), diff(i(t), t) = 1· s(t)·i(t) − 4·i(t)}
```

$$sys := \left\{ \frac{d}{dt}\, i(t) = s(t)\, i(t) - 4\, i(t),\ \frac{d}{dt}\, s(t) = -s(t)\, i(t) \right\} \quad (1)$$

```
> DEplot(sys, [s(t), i(t)], t = 0..5, [[s(0) = 9.9, i(0) = 0.1], [s(0) = 7.0, i(0) = 0.1],
    [s(0) = 3.0, i(0) = 7.0]], s = 0..10, i = 0..10, linecolor = [blue, green, brown],
    numpoints = 1000)
```



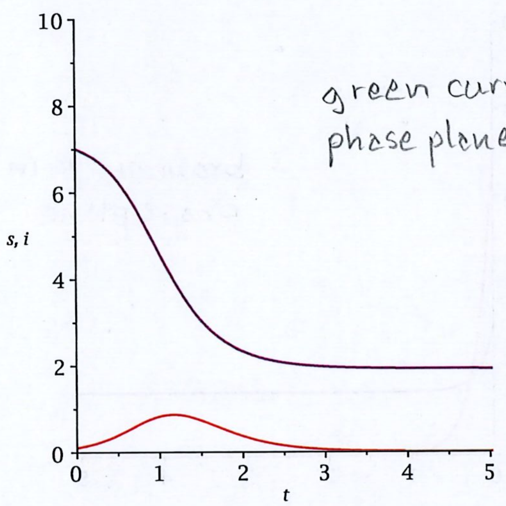

```
>
p1 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(i(t), t) = 1· s(t)·i(t) − 4·i(t),
    s(0) = 9.9, i(0) = 0.1}, {s(t), i(t)}, type = numeric, range = 0..500) :
> p2 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(i(t), t) = 1· s(t)·i(t) − 4·i(t), s(0)
    = 7.0, i(0) = 0.1}, {s(t), i(t)}, type = numeric, range = 0..500) :
  with(plots) :
  p3 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(i(t), t) = 1· s(t)·i(t) − 4·i(t),
    s(0) = 3.0, i(0) = 7.0}, {s(t), i(t)}, type = numeric, range = 0..500) :
  with(plots) :

> odeplot(p1, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
    ..10], refine = 1)
```



purple = S(t)
orange = I(t)

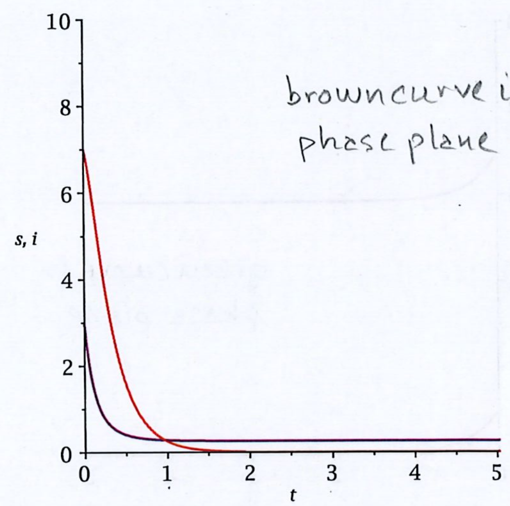
blue curve in phase plane

```
> odeplot(p2, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
    ..10], refine = 1)
```



green curve in phase plane

```
> odeplot(p3, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
    ..10], refine = 1)
```

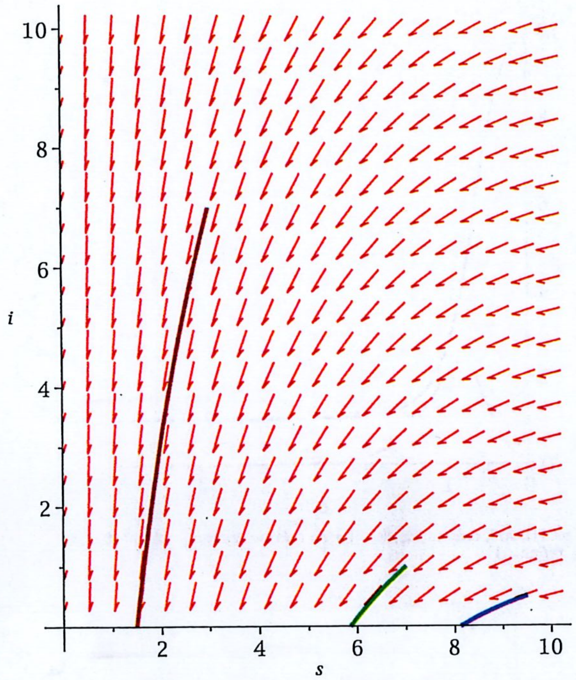


browncurve in phase plane

```
>

  sys2 := {diff(s(t), t) = -0.5· s(t)·i(t), diff(i(t), t) = 0.5· s(t)·i(t) − 6·i(t)} :
> DEplot(sys2, [s(t), i(t)], t = 0..5, [[s(0) = 9.5, i(0) = 0.5], [s(0) = 7.0, i(0)
    = 1.0], [s(0) = 3.0, i(0) = 7.0]], s = 0..10, i = 0..10, linecolor = [blue, green,
    brown], numpoints = 1000)
```

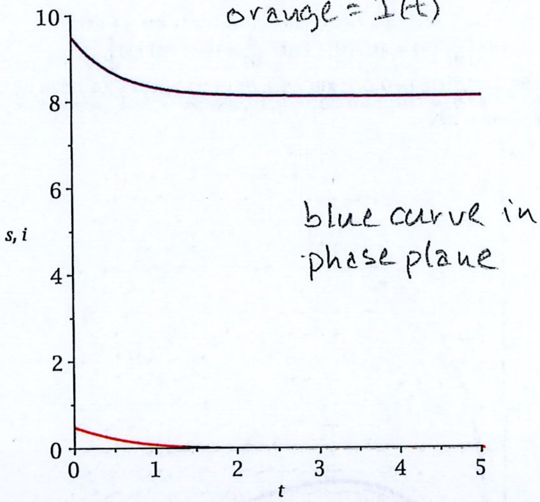
11.6

SIR model, $N=10$, $\beta=0.5$, $\nu=6$ $\Rightarrow$ $\frac{\nu}{\beta} > N$ (case 2) and $R_0 = \frac{5}{6} < 1$

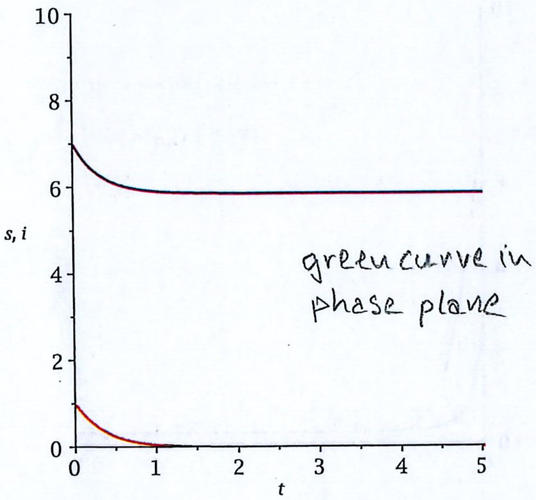



```
> p4 := dsolve({diff(s(t), t) = -0.5· s(t)·i(t), diff(i(t), t) = 0.5· s(t)·i(t) - 6·i(t),
    s(0) = 9.5, i(0) = 0.5}, {s(t), i(t)}, type = numeric, range = 0..500) :
> p5 := dsolve({diff(s(t), t) = -0.5· s(t)·i(t), diff(i(t), t) = 0.5· s(t)·i(t) - 6·i(t),
    s(0) = 7.0, i(0) = 1.0}, {s(t), i(t)}, type = numeric, range = 0..500) :
> p6 := dsolve({diff(s(t), t) = -0.5· s(t)·i(t), diff(i(t), t) = 0.5· s(t)·i(t) - 6·i(t),
    s(0) = 3.0, i(0) = 7.0}, {s(t), i(t)}, type = numeric, range = 0..500) :
> odeplot(p4, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
    ..10], refine = 1)
```

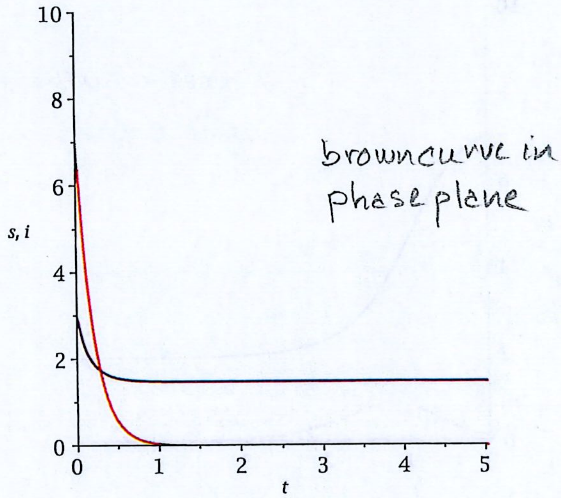
purple = S(t)
orange = I(t)




```
> odeplot(p5, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
    ..10], refine = 1)
```




```
> odeplot(p6, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
    ..10], refine = 1)
```



>

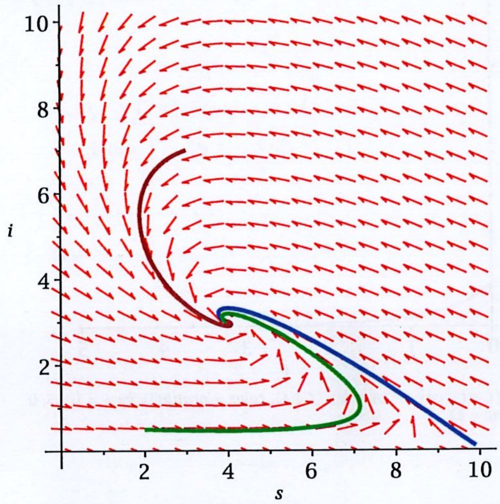
SIRS model, $N=10, \beta=1, \nu=4, \gamma=4 \Rightarrow \frac{\nu}{\beta} < N$ (case 1) and $R_0 = 2.5 > 1$

steady state $(\bar{S}, \bar{I}) = (4, 3)$ (stable)

```
> with(DEtools) :
> sys := {diff(s(t), t) = 4·(10 − s(t) − i(t))−1· s(t)·i(t), diff(i(t), t) = 1· s(t)·i(t)
     − 4·i(t)}
```

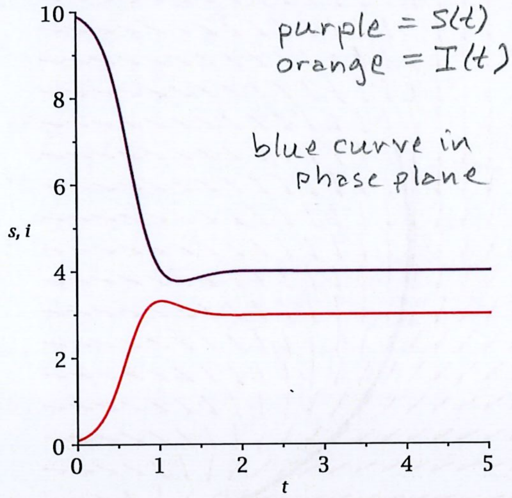
$$sys := \left\{ \frac{d}{dt} i(t) = s(t)\, i(t) - 4\, i(t),\ \frac{d}{dt} s(t) = 40 - 4\, s(t) - 4\, i(t) - s(t)\, i(t) \right\} \quad (1)$$

```
> DEplot(sys, [s(t), i(t)], t = 0..5, [[s(0) = 9.9, i(0) = 0.1], [s(0) = 2.0, i(0) = 0.5],
     [s(0) = 3.0, i(0) = 7.0]], s = 0..10, i = 0..10, linecolor = [blue, green, brown],
     numpoints = 1000)
```

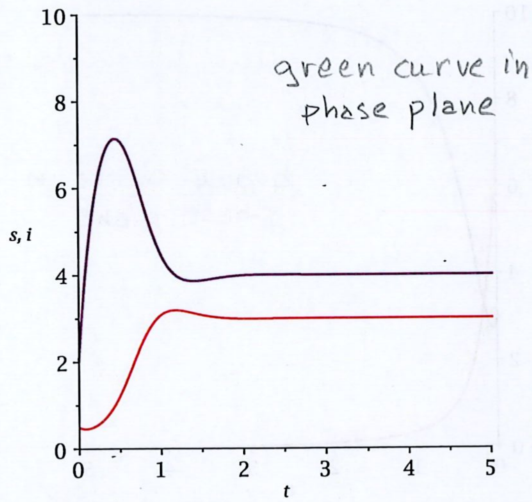



```
>
p1 := dsolve({diff(s(t), t) = 4·(10 − s(t) − i(t))−1· s(t)·i(t), diff(i(t), t) = 1
     · s(t)·i(t) − 4·i(t), s(0) = 9.9, i(0) = 0.1}, {s(t), i(t)}, type = numeric, range
     = 0..500) :
> p2 := dsolve({diff(s(t), t) = 4·(10 − s(t) − i(t))−1· s(t)·i(t), diff(i(t), t) = 1
     · s(t)·i(t) − 4·i(t), s(0) = 2.0, i(0) = 0.5}, {s(t), i(t)}, type = numeric, range
     = 0..500) : with(plots) :
  p3 := dsolve({diff(s(t), t) = 4·(10 − s(t) − i(t))−1· s(t)·i(t), diff(i(t), t) = 1
     · s(t)·i(t) − 4·i(t), s(0) = 3.0, i(0) = 7.0}, {s(t), i(t)}, type = numeric, range
     = 0..500) : with(plots) :
```

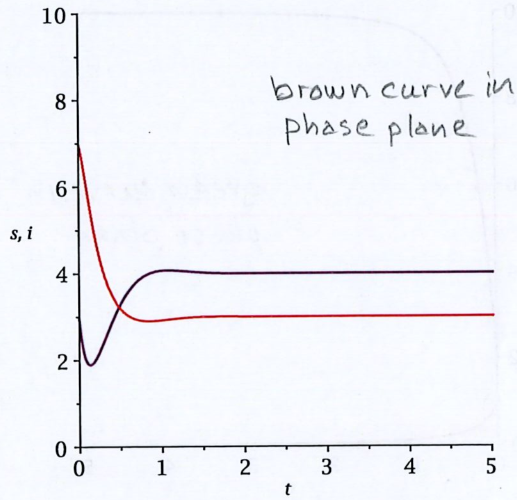
```
> odeplot(p1, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
     ..10], refine = 1)
```




```
> odeplot(p2, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
     ..10], refine = 1)
```



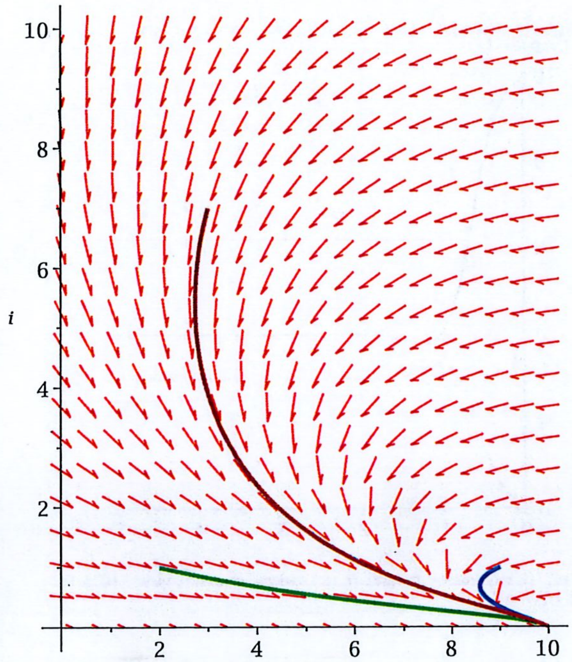

```
> odeplot(p3, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0
     ..10], refine = 1)
```




```
>
sys2 := {diff(s(t), t) = 4·(10 − s(t) − i(t))−0.5· s(t)·i(t), diff(i(t), t) = 0.5
     · s(t)·i(t) − 6·i(t)} :
> DEplot(sys2, [s(t), i(t)], t = 0..5, [[s(0) = 9.0, i(0) = 1.0], [s(0) = 2.0, i(0)
     = 1.0], [s(0) = 3.0, i(0) = 7.0]], s = 0..10, i = 0..10, linecolor = [blue, green,
     brown], numpoints = 1000)
```

SIRS model $N=10$, $\beta=0.5$, $\nu=6$, $\gamma=4$ $\Rightarrow \frac{\nu}{\beta} > N$ (case 2) and $R_0 = \frac{5}{6} < 1$

Stable steady state is $(10,0)$

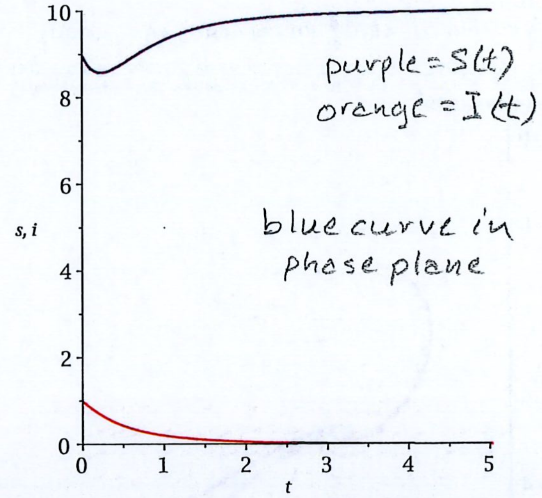


> p4 := dsolve({diff(s(t), t) = 4·(10 − s(t) − i(t))−0.5· s(t)·i(t), diff(i(t), t) = 0.5 · s(t)·i(t) − 6·i(t), s(0) = 9.0, i(0) = 1.0}, {s(t), i(t)}, type = numeric, range = 0..500) :

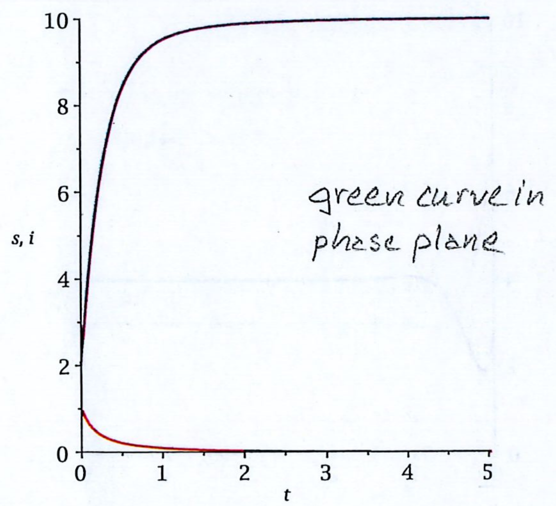
> p5 := dsolve({diff(s(t), t) = 4·(10 − s(t) − i(t))−0.5· s(t)·i(t), diff(i(t), t) = 0.5 · s(t)·i(t) − 6·i(t), s(0) = 2.0, i(0) = 1.0}, {s(t), i(t)}, type = numeric, range = 0..500) :

> p6 := dsolve({diff(s(t), t) = 4·(10 − s(t) − i(t))−0.5· s(t)·i(t), diff(i(t), t) = 0.5 · s(t)·i(t) − 6·i(t), s(0) = 3.0, i(0) = 7.0}, {s(t), i(t)}, type = numeric, range = 0..500) :

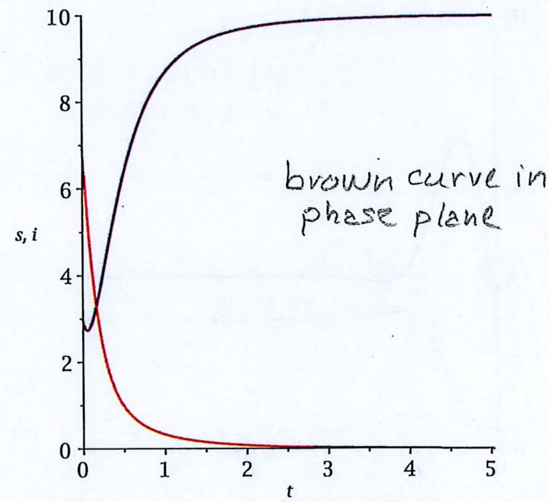
> odeplot(p4, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0..10], refine = 1)



> odeplot(p5, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0..10], refine = 1)



> odeplot(p6, [[t, s(t), color = purple], [t, i(t), color = orange]], view = [0..5, 0..10], refine = 1)



>

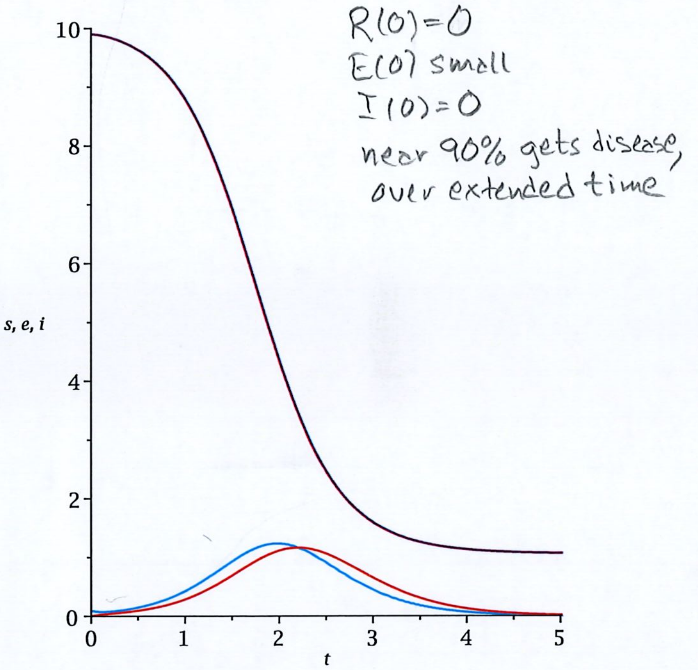
SEIR model , $N=10$, $\beta=1$, $\delta=4$, $\nu=4$ $\Rightarrow$ $\frac{\nu}{\beta}<N$, $R_0=2.5>1$ | 11.9

$\begin{cases} \text{purple} = S(t) \\ \text{blue} = E(t) \to 0,\ t\to\infty \\ \text{orange} = I(t) \to 0,\ t\to\infty \end{cases}$

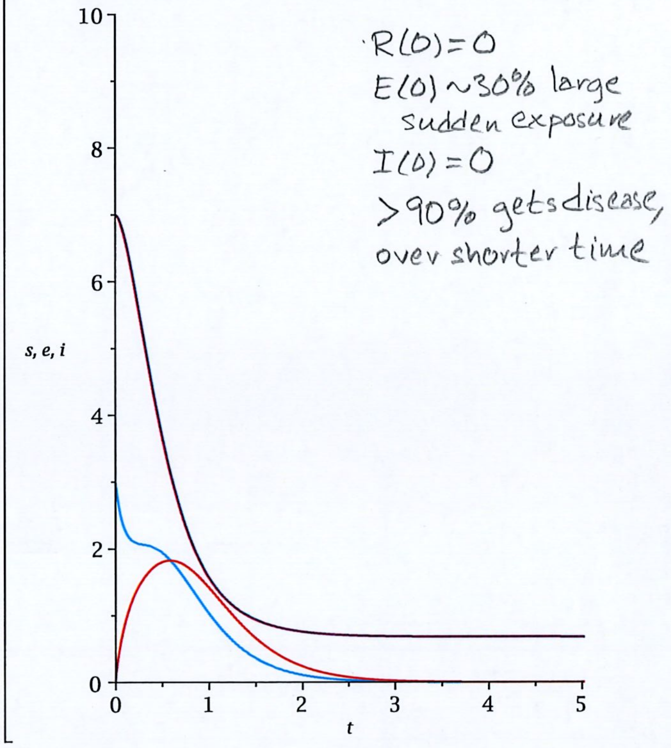
$R(t)=10-S(t)-E(t)-I(t)$, $R(\infty)=10-S(\infty)$

$R(0)$ = initial immunity , $R(\infty)-R(0)$ = number of disease cases during epidemy

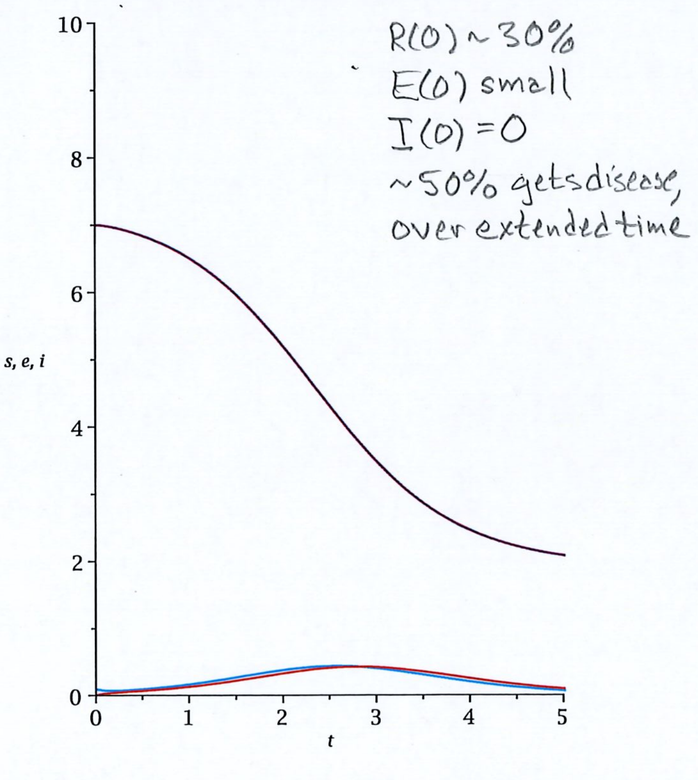
```
> with(DEtools) :
> with(plots) :
>
  p1 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(e(t), t) = 1· s(t)·i(t) - 4·e(t),
     diff(i(t), t) = 4· e(t) - 4·i(t), s(0) = 9.9, e(0) = 0.1, i(0) = 0}, {s(t), e(t),
     i(t)}, type = numeric, range = 0..500) :
> odeplot(p1, [[t, s(t), color = purple], [t, e(t), color = cyan], [t, i(t), color
     = orange]], view = [0..5, 0..10], refine = 1)
```



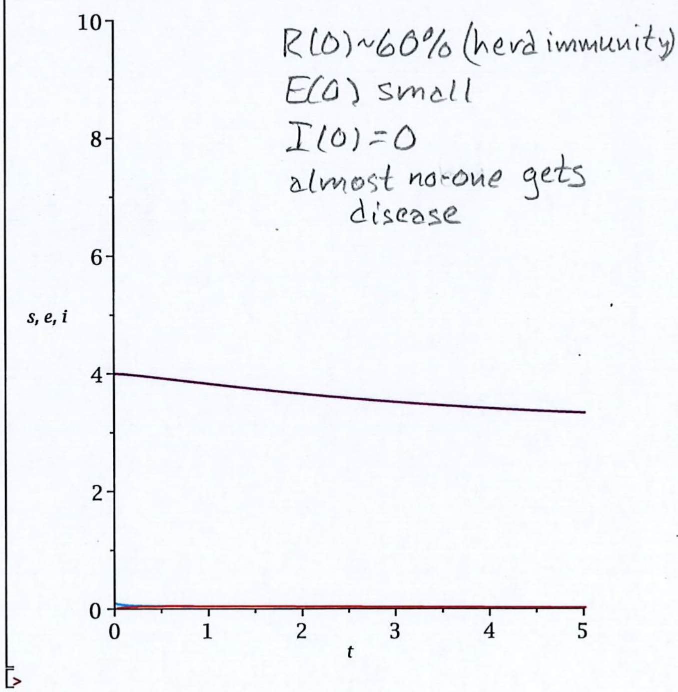

```
> p2 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(e(t), t) = 1· s(t)·i(t) - 4·e(t),
     diff(i(t), t) = 4· e(t) - 4·i(t), s(0) = 7, e(0) = 3, i(0) = 0}, {s(t), e(t), i(t)},
     type = numeric, range = 0..500) :
> odeplot(p2, [[t, s(t), color = purple], [t, e(t), color = cyan], [t, i(t), color
     = orange]], view = [0..5, 0..10], refine = 1)
```




```
> p3 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(e(t), t) = 1· s(t)·i(t) - 4·e(t),
     diff(i(t), t) = 4· e(t) - 4·i(t), s(0) = 7.0, e(0) = 0.1, i(0) = 0}, {s(t), e(t),
     i(t)}, type = numeric, range = 0..500) :
> odeplot(p3, [[t, s(t), color = purple], [t, e(t), color = cyan], [t, i(t), color
     = orange]], view = [0..5, 0..10], refine = 1)
```




```
> p4 := dsolve({diff(s(t), t) = -1· s(t)·i(t), diff(e(t), t) = 1· s(t)·i(t) - 4·e(t),
     diff(i(t), t) = 4· e(t) - 4·i(t), s(0) = 4, e(0) = 0.1, i(0) = 0.0}, {s(t), e(t),
     i(t)}, type = numeric, range = 0..500) :
> odeplot(p4, [[t, s(t), color = purple], [t, e(t), color = cyan], [t, i(t), color
     = orange]], view = [0..5, 0..10], refine = 1)
```




```
>
```