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Linear difference equations of order 1 and 2

$f(n)$, $g(n)$, $h(n)$ given functions of n, where $n \geq 0$ is an integer. Find the function (sequence) $x(n) = x_n$ such that

order 1: $\quad x_{n+1} + f(n)x_n = g(n) \qquad (1)$

order 2: $\quad x_{n+2} + f(n)x_{n+1} + g(n)x_n = h(n) \qquad (2)$

Both have solution structure $x_n = x_{n,h} + x_{n,p}$, where $x_{n,h}$ are all homogeneous solutions (the solutions if $g(n) = 0$ in (1) and $h(n) = 0$ in (2)), $x_{n,p}$ is one particular solution.

These are more difficult to solve than the corresponding differential equations because there are no primitive functions. For constant coefficients the homogeneous solution is easy to find:

$x_{n+1} + ax_n = 0$ has the solution $x_n = c(-a)^n$ (3).

$x_{n+2} + ax_{n+1} + bx_n = 0$ has the solution $x_n = c_1(r_1)^n + c_2(r_2)^n$ (4) ,

where $r_1 \neq r_2$ are the solutions to $r^2 + ar + b = 0$. In case $r_1 = r_2$, then $x_n = (c_1 n + c_2)r_1^n$. If $r_{1,2} = \alpha \pm i\beta = \rho e^{\pm i\varphi}$ (polar form) are complex, one can write $r_{1,2}^n = \rho^n e^{\pm in\varphi}$.

The constant c in (3) can be determined by an initial value of x_0. In (4) c_1 and c_2 are determined by giving x_0 and x_1.

For non-homogeneous equations, a particular solution is found by some Ansatz.

Observe that the condition for $x_n \to 0$ as $n \to \infty$ in (3) is $|a| < 1$, and in (4) $|r_{1,2}| < 1$.

<u>Example</u>

Solve $x_{n+2} - x_{n+1} - 6x_n = 0$ with initial conditions $x_0 = 3$ and $x_1 = 4$

Solution: the solutions to $r^2 - r - 6 = 0$ are $r_1 = 3$ and $r_2 = -2$ $\Rightarrow$ the general solution is $x_n = c_1 3^n + c_2(-2)^n$

The initial condition at $n = 0$ gives $x_0 = c_1 3^0 + c_2(-2)^0 = c_1 + c_2 = 3$ (1), and at $n = 1$ we get $x_1 = c_1 3^1 + c_2(-2)^1 = 3c_1 - 2c_2 = 4$ (2). (1) and (2) $\Rightarrow$ $c_1 = 2$ and $c_2 = 1$ $\Rightarrow$ the solution with the given initial conditions is $x_n = 2 \cdot 3^n + (-2)^n$

TEST QUESTIONS

1. Solve $x_{n+1} + 3x_n = 0$ with initial condition $x_0 = 5$
2. Solve $x_{n+2} + 6x_{n+1} + 5x_n = 0$ with initial conditions $x_0 = 5$ and $x_1 = 3$
3. Solve $x_{n+2} + 6x_{n+1} + 9x_n = 0$ with initial conditions $x_0 = 5$ and $x_1 = 3$
4. Solve $x_{n+2} - 2x_{n+1} + 2x_n = 0$ with initial conditions $x_0 = 5$ and $x_1 = 3$

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ANSWERS

1. General solution is $x_n = c(-3)^n$.
Initial condition $x_0 = 5 \Rightarrow c = 5 \Rightarrow x_n = 5(-3)^n$.

2. General solution is $x_n = c_1(-1)^n + c_2(-5)^n$.
Initial conditions $x_0 = 5,\ x_1 = 3 \ \Rightarrow\ c_1 + c_2 = 5,\ -c_1 - 5c_2 = 3 \Rightarrow c_1 = 7,\ c_2 = -2 \Rightarrow$
$x_n = 7(-1)^n - 2(-5)^n$

3. General solution is $x_n = (c_1 n + c_2)(-3)^n$
Initial conditions $x_0 = 5,\ x_1 = 3 \Rightarrow c_2 = 5,\ (c_1 + c_2)(-3) = 3 \Rightarrow c_1 = -6,\ c_2 = 5 \Rightarrow$
$x_n = (-6n + 5)(-3)^n$

4. General solution is $x_n = c_1(1+i)^n + c_2(1-i)^n = (\sqrt{2})^n(c_1 e^{in\pi/4} + c_2 e^{-in\pi/4}) =$
$= 2^{n/2}(c_3 \cos\frac{n\pi}{4} + c_4 \sin\frac{n\pi}{4})$
Initial conditions $x_0 = 5,\ x_1 = 3 \ \Rightarrow\ c_3 = 5,\ 2^{1/2}(c_3\frac{1}{\sqrt{2}} + c_4\frac{1}{\sqrt{2}}) = 3 \Rightarrow c_3 = 5,\ c_4 = -2 \Rightarrow$
$x_n = 2^{n/2}(5\cos\frac{n\pi}{4} - 2\sin\frac{n\pi}{4})$

Time discrete models

[First: quick view of linear difference equations]

Cell division

If cells divide synchronously at times labelled $n = 0, 1, 2, \dots$ such that each cell produces a new cells, then, if M_n denotes the number of cells at time n, we have

$$M_1 = aM_0 \,,\; M_2 = aM_1 = a^2 M_0 \,, \dots,\; M_n = aM_{n-1} = \dots = a^n M_0$$

$M_n = aM_{n-1}$ is called a difference equation (or recursion or iteration formula) of order 1. The solution is $M_n = a^n M_0$.

Note $M_n \to \begin{cases} \infty & \text{if } a > 1 \\ M_0 & \text{if } a = 1 \\ 0 & \text{if } 0 \le a < 1 \end{cases}$ ($a \geq 0$ for the cells)

Fibonacci's rabbits

Suppose rabbits reproduce twice, at the age of one and two months, and that each pair has one pair of newborns (one male, one female). Suppose all survive ≥ 2 months. Let R_n be the number of newborn pairs in the n^{th} generation, and suppose $R_0 = 1$.

$R_0 = 1 \to R_1 = 1 \to R_2 = 2 \to R_3 = 3 \to R_4 = 5 \to R_5 = 8 \; \dots$ etc $13, 21, 34, 55, \dots$

We have $R_{n+2} = R_{n+1} + R_n$, a difference eq. of order 2

Solve $R_{n+2} - R_{n+1} - R_n = 0$ with initial values $R_0 = 1, R_1 = 1$

$$r^2 - r - 1 = 0 \Rightarrow r_{1,2} = \frac{1 \pm \sqrt{5}}{2} \Rightarrow R_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\left.\begin{array}{l} R_0 = c_1 + c_2 = 1 \\ R_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} = 1 \end{array}\right\} \Rightarrow \dots \Rightarrow c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \,,\; c_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}} \Rightarrow$$

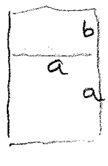
$$\Rightarrow R_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$ (which is an integer!)

Note: $|r_2| = \left|\frac{1-\sqrt{5}}{2}\right| < 1 \Rightarrow \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \to 0,\; n \to \infty$, and

$R_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ if n large. This is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.62$

R_n increases by $\sim 62\%$ each generation.

φ is common in art and science, see page 4 in EK.

(Sunflower spirals) [If $\frac{a}{b} = \frac{a+b}{a}$ then $\varphi = \frac{a}{b} = 1 + \frac{1}{\varphi}$ ]

Propagation of annual plants

P_n = number of plants year n
γ = number of seeds produced by each plant in August
σ = fraction of seeds that survives winter (seeds can survive two winters)
α = fraction germinating after one year
β = fraction germinating after two years

Find equation for P_n and solve it.

Let $S_n^A = \gamma P_n$ = number of seeds in August

$S_{n+1}^M = \sigma S_n^A = \sigma\gamma P_n$ = seeds in May next year (survived winter), gives

$\alpha S_{n+1}^M = \alpha\sigma\gamma P_n$ new flowers

$(1-\alpha)S_{n+1}^M$ left, $\sigma(1-\alpha)S_{n+1}^M$ survives 2^{nd} winter, gives $\beta\sigma(1-\alpha)S_{n+1}^M$ new flowers year 2. Shifted one year, year 1 gets $\beta\sigma(1-\alpha)\underbrace{S_n^M}_{\sigma\gamma P_{n-1}}$ new flowers from two-year old seeds.

In total, flowers year $n+1$:

$$P_{n+1} = \underbrace{\alpha\sigma\gamma P_n}_{\text{1-year old seeds}} + \underbrace{\beta\sigma^2(1-\alpha)\gamma P_{n-1}}_{\text{2-year old seeds}}$$

linear difference eq. order 2
constant coefficients

$$P_{n+2} - \underbrace{\alpha\sigma\gamma}_{a>0} P_{n+1} - \underbrace{\beta\sigma^2(1-\alpha)\gamma}_{b>0} P_n = 0 \qquad r^2 - ar - b = 0 \Rightarrow$$

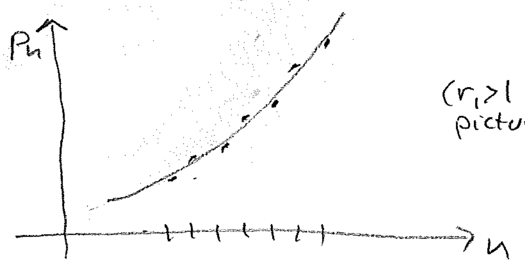
$r_{1,2} = \frac{a}{2} \pm \frac{1}{2}\underbrace{\sqrt{a^2+4b}}_{>a} \Rightarrow r_1 = \frac{a+\sqrt{a^2+4b}}{2} > 0$ and $r_2 = \frac{a-\sqrt{a^2+4b}}{2} < 0$ both real

and $r_1 > |r_2|$

The solution is $P_n = c_1 r_1^n + c_2 r_2^n$

For large n, $c_1 r_1^n$ dominates, $c_2 r_2^n$ has alternating signs $\Rightarrow$

$P_n \approx c_1 r_1^n$ + smaller oscillating part



($r_1 > 1$ in picture)

In original units

$$r_1 = \frac{\alpha\sigma\gamma}{2}\left(1+\sqrt{1+\frac{4\beta(1-\alpha)}{\alpha^2\gamma}}\right)$$

Suppose β small $\Rightarrow r_1 \approx \alpha\sigma\gamma$

For survival we need $r_1 \gtrsim 1$ (otherwise $P_n \to 0$) $\Rightarrow \alpha\sigma\gamma \gtrsim 1$
natural condition!

If $\underbrace{\sigma = \frac{1}{2}}_{\text{50\% survive winter}}$ and $\underbrace{\alpha = \frac{1}{5}}_{\text{20\% germinate}}$, only 10% of seeds result in a new plant

next year $\Rightarrow$ one needs $\gamma \gtrsim 10$ seeds from each plant (this is if β small)