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Linear systems of 2 (order-1) difference equations (constant coefficients)

Find the sequences x_n and y_n such that $\begin{cases} x_{n+1} = ax_n + by_n \\ y_{n+1} = cx_n + dy_n \end{cases}$ $(a, b, c, d$ constants)

With matrices: $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=A} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

Eigenvalues and eigenvectors of A: $A\bar{v}_1 = \lambda_1\bar{v}_1$, $A\bar{v}_2 = \lambda_2\bar{v}_2$ (assume two independent eigenvectors exist even if $\lambda_1 = \lambda_2$)

Diagonalization of A: $A = TDT^{-1}$ with $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $\bar{v}_1$, $\bar{v}_2$ columns of T.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = TDT^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \Rightarrow T^{-1}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = DT^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}. \text{ Put } T^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = D\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = \begin{pmatrix} \lambda_1\tilde{x}_n \\ \lambda_2\tilde{y}_n \end{pmatrix} \Rightarrow \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^n \\ c_2\lambda_2^n \end{pmatrix} \Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = T\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = \tilde{x}_n\bar{v}_1 + \tilde{y}_n\bar{v}_2$$

Therefore, all solutions to the system are $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1\lambda_1^n\bar{v}_1 + c_2\lambda_2^n\bar{v}_2$

In the case of complex eigenvalues and eigenvectors, $\lambda_{1,2} = \alpha \pm i\beta = \rho e^{\pm i\varphi}$, $\bar{v}_{1,2} = \bar{u} \pm i\bar{w}$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \rho^n[c_1e^{in\varphi}(\bar{u} + i\bar{w}) + c_2e^{-in\varphi}(\bar{u} - i\bar{w})] =$$

$$= \rho^n[c_1(\cos n\varphi + i\sin n\varphi)(\bar{u} + i\bar{w}) + c_2(\cos n\varphi - i\sin n\varphi)(\bar{u} - i\bar{w})] =$$

$$= \underbrace{\rho^n}_{\text{expon.}} \underbrace{[\overbrace{(c_1 + c_2)}^{k_1}(\bar{u}\cos n\varphi - \bar{w}\sin n\varphi) + \overbrace{i(c_1 - c_2)}^{k_2}(\bar{u}\sin n\varphi + \bar{w}\cos n\varphi)]}_{\text{"oscillating"}}, \text{ real expression if } k_1, k_2 \text{ real}$$

Asymptotic (long-time) behaviour of solutions (as $n \to \infty$):

If all eigenvalues $|\lambda_j| < 1$ then $\lambda_j^n \to 0$ as $n \to \infty$ for all j $\Rightarrow$ solutions $\to 0$

If some $|\lambda_j| > 1$ then the magnitude of solutions grow (to ∞ as $n \to \infty$)

Systems of 2 difference equations and single order-2 difference equations:

Note that $x_{n+1} = ax_n + by_n \Rightarrow x_{n+2} = ax_{n+1} + by_{n+1} = ax_{n+1} + b(cx_n + dy_n) = ax_{n+1} + bcx_n + d(x_{n+1} - ax_n) = (a + d)x_{n+1} + (bc - ad)x_n \Leftrightarrow x_{n+2} - (a + d)x_{n+1} + (ad - bc)x_n = 0$. The system therefore gives this order-2 difference equation with constant coefficients for x_n. After solving this for x_n, y_n is determined (if $b \neq 0$) from $by_n = x_{n+1} - ax_n$.

Conversely, defining $y_n = x_{n+1}$, an order-2 difference equation $x_{n+2} + ax_{n+1} + bx_n = 0$ can be written as an order-1 system $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}}_{=A} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

Example

Solve $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$.

Solve also the system by rewriting it as an order-2 difference equation for x_n.

Solution:

1. Eigenvalues of $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Solutions are $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 3^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(-2)^n \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

Finding the order-2 difference equation for x_n:

$x_{n+1} = 2x_n + y_n$, $y_{n+1} = 4x_n - y_n \Rightarrow x_{n+2} = 2x_{n+1} + y_{n+1} = 2x_{n+1} + (4x_n - y_n) =$
$= 2x_{n+1} + 4x_n - (x_{n+1} - 2x_n) = x_{n+1} + 6x_n \Rightarrow$
$x_{n+2} - x_{n+1} - 6x_n = 0$, which has solution $x_n = c_1 3^n + c_2(-2)^n$.
Then $y_n = x_{n+1} - 2x_n = 3c_1 3^n - 2c_2(-2)^n - 2(c_1 3^n + c_2(-2)^n) = c_1 3^n - 4c_2(-2)^n$

Modeling with systems of difference equations

The model for plants (sem.12) can be written as a 1st-order system. If we also use s_n^M (seeds in May) as an unknown, we have

$$\begin{cases} P_{n+1} = \alpha\sigma\gamma P_n + \beta\sigma(1-\alpha)s_n^M \\ s_{n+1}^M = \sigma\gamma P_n \end{cases}$$

Can be solved with eigenvalues and eigenvectors. Eigenvalues are $\lambda_{1,2} = r_{1,2}$!

Another model. (Ek page 27 and problem 1.16):

Red blood cell (RBC) production

R_n = number of RBC's in circulation, day n

M_n = — " — produced by marrow, day n [sv: märg]

f = fraction of RBC's removed by spleen, $0<f<1$ [sv: mjälte]

γ = number of RBC's produced day $n+1$ per RBC lost day n

Model:

$$\begin{cases} R_{n+1} = R_n - \overbrace{fR_n}^{\text{removed}} + \overbrace{M_n}^{\text{new}} = (1-f)R_n + M_n \\ M_{n+1} = \gamma \cdot \underbrace{fR_n}_{\text{removed}} \end{cases}$$

Goal: Need R_n balanced ($\sim$ constant)

We could extract a single order-2 equation for R_n but we solve as a system.

$$\begin{pmatrix} R_{n+1} \\ M_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1-f & 1 \\ \gamma f & 0 \end{pmatrix}}_{A} \begin{pmatrix} R_n \\ M_n \end{pmatrix}$$

Eigenvalues $\begin{vmatrix} 1-f-\lambda & 1 \\ \gamma f & -\lambda \end{vmatrix} = \lambda^2 - (1-f)\lambda - \gamma f = 0 \Rightarrow$

$$\lambda_{1,2} = \underbrace{\frac{1-f}{2}}_{>0} \pm \frac{1}{2}\underbrace{\sqrt{(1-f)^2 + 4\gamma f}}_{>1-f}$$, real with $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_1 > |\lambda_2|$

corresponding eigenvectors are $\bar{v}_{1,2} = \begin{pmatrix} 1 \\ -\frac{1-f}{2} \pm \frac{1}{2}\sqrt{(1-f)^2 + 4\gamma f} \end{pmatrix}$

Solutions become

$$\begin{pmatrix} R_n \\ M_n \end{pmatrix} = c_1 \lambda_1^n \bar{v}_1 + c_2 \lambda_2^n \bar{v}_2 .$$

For n large $\begin{pmatrix} R_n \\ M_n \end{pmatrix} \approx c_1 \lambda_1^n \bar{v}_1 \Rightarrow R_n \approx c_1 \lambda_1^n$

$\lambda_1 \approx 1$ needed for balanced level of RBC $\Rightarrow$

$$\frac{1-f}{2} + \frac{1}{2}\sqrt{(1-f)^2+4\gamma f} = 1 \Rightarrow \sqrt{(1-f)^2+4\gamma f} = 1+f \Rightarrow (1-f)^2+4\gamma f = (1+f)^2$$

$\Rightarrow 4\gamma f = 4f \Rightarrow \gamma \approx 1$ needed (natural!)

$\lambda_1 + \lambda_2 = \mathrm{Tr}\,A = 1-f$ so $\lambda_1 \approx 1 \Rightarrow \lambda_2 \approx -f \Rightarrow$

$$R_n \approx c_1 \cdot 1^n + c_2(-f)^n = c_1 + c_2(-f)^n$$

($|c_2| \ll c_1$ for realistic start)