Non-linear difference equations of order 1

$x_{n+1} = f(x_n)$, f not linear in general

Example $x_{n+1} = r x_n(1-x_n)$, $r>0$ constant , the logistic (difference) equation

$\bar{x}$ is a steady state (equilibrium, fixed point, critical point) if
$x_n = \bar{x} \Rightarrow x_{n+1} = \bar{x}$, so $f(\bar{x}) = \bar{x}$ ($\bar{x}$ fixed point of f).

Stability of $\bar{x}$

If x_n is near $\bar{x}$, let $\tilde{x}_n = x_n - \bar{x}$ (small) $\Rightarrow$

$$\tilde{x}_{n+1} = x_{n+1} - \bar{x} = f(x_n) - f(\bar{x}) = f(\bar{x} + \tilde{x}_n) - f(\bar{x}) = \Big/ \begin{array}{c}\text{Taylor expansion}\\ \text{of } f \text{ at } \bar{x}\end{array} \Big/ =$$

$$= f(\bar{x}) + \underbrace{f'(\bar{x})}_{=a}\tilde{x}_n + O(\tilde{x}_n^2) - f(\bar{x}) \approx a\tilde{x}_n \quad \text{(linearization)}$$

$\Rightarrow \tilde{x}_n \approx c \cdot a^n$ and $\tilde{x}_n \to 0, n \to \infty \Leftrightarrow |a| < 1$

$\tilde{x}_n \to 0 \Leftrightarrow x_n \to \bar{x}$, so $\begin{cases} \bar{x} \text{ is stable if } |f'(\bar{x})| < 1 \\ \bar{x} \text{ is unstable if } |f'(x)| > 1 \end{cases}$

($|f'(\bar{x})| = 1$ is a limit case)

Ex Logistic eq.

$$x_{n+1} = f(x_n) = r x_n(1-x_n) \quad , r > 0$$

Steady states : $\bar{x} = r\bar{x}(1-\bar{x}) \Leftrightarrow \bar{x}(1-r+r\bar{x}) = 0$

$\Rightarrow \bar{x}_1 = 0$ or $\bar{x}_2 = \frac{r-1}{r} = 1 - \frac{1}{r}$. Check stability

$f(x) = r(x-x^2) \Rightarrow f'(x) = r(1-2x) \Rightarrow$

$|f'(0)| = |r| = r$, stable if $0 < r < 1$ ($r>0$ always)

$|f'(1-\frac{1}{r})| = |r(1-2+\frac{2}{r})| = |2-r|$, stable if $|2-r| < 1 \Leftrightarrow 1 < r < 3$

No stable steady states if $r > 3$.

Much more on logistic eq. next seminar

Non-linear systems (time discrete dynamical system)

$$\begin{cases} x_{n+1} = f_1(x_n, y_n) \\ y_{n+1} = f_2(x_n, y_n) \end{cases} \qquad n = 0, 1, 2, \ldots$$

$(\bar{x}, \bar{y})$ is a steady state (fixedpoint, ...) if $\begin{cases} \bar{x} = f_1(\bar{x}, \bar{y}) \\ \bar{y} = f_2(\bar{x}, \bar{y}) \end{cases}$

$$\left((x_n, y_n) = (\bar{x}, \bar{y}) \Rightarrow (x_{n+1}, y_{n+1}) = (\bar{x}, \bar{y}) \right)$$

In the same way as for single equations, if $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ near $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, let

$$\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \Rightarrow \begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} =$$

$$= \begin{pmatrix} f_1(\bar{x} + \tilde{x}_n, \bar{y} + \tilde{y}_n) \\ f_2(\bar{x} + \tilde{x}_n, \bar{y} + \tilde{y}_n) \end{pmatrix} - \begin{pmatrix} f_1(\bar{x}, \bar{y}) \\ f_2(\bar{x}, \bar{y}) \end{pmatrix} = / \text{Taylor of } f_1 \text{ and } f_2 \text{ at } \bar{x}, \bar{y} / =$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x}(\bar{x}, \bar{y}) \tilde{x}_n + \frac{\partial f_1}{\partial y}(\bar{x}, \bar{y}) \tilde{y}_n + O(\tilde{x}_n^2 + \tilde{y}_n^2) \\ \frac{\partial f_2}{\partial x}(\bar{x}, \bar{y}) \tilde{x}_n + \frac{\partial f_2}{\partial y}(\bar{x}, \bar{y}) \tilde{y}_n + O(\tilde{x}_n^2 + \tilde{y}_n^2) \end{pmatrix} \approx \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial f_1}{\partial y}(\bar{x}, \bar{y}) \\ \frac{\partial f_2}{\partial x}(\bar{x}, \bar{y}) & \frac{\partial f_2}{\partial y}(\bar{x}, \bar{y}) \end{pmatrix}}_{J(\bar{x}, \bar{y})} \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}$$

linearization at $(\bar{x}, \bar{y})$

If λ_1 and λ_2 are the eigenvalues of $J(\bar{x}, \bar{y})$, $\tilde{x}_n$ and $\tilde{y}_n$ have terms with λ_1^n and λ_2^n. We get that $(\bar{x}, \bar{y})$ is stable ($(\tilde{x}_n, \tilde{y}_n) \to (0,0)$ or $(x_n, y_n) \to (\bar{x}, \bar{y})$) if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

With $J = J(\bar{x}, \bar{y})$, $\lambda_1 + \lambda_2 = \operatorname{Tr} J$ and $\lambda_1 \lambda_2 = \det J$.

One can show (Jury test, see EK 2.8) that

$$|\lambda_{1,2}| < 1 \iff |\operatorname{Tr} J| < 1 + \det J < 2 \qquad \text{(sometimes useful)}$$

Example Time discrete SIR model

S_n = number of susceptible, time n (scale is often days)

I_n = number of infective

R_n = number of removed

$S_n + I_n + R_n = N$ = total population (constant)

$$\begin{cases} S_{n+1} = S_n - \beta S_n I_n \\ I_{n+1} = I_n + \beta S_n I_n - \nu I_n \\ R_{n+1} = R_n + \nu I_n \end{cases}$$

we can study the 2D-system for S_n and I_n. Then $R_n = N - S_n - I_n$

$R_0 = \frac{N\beta}{\nu}$ = basic reproduction number [not the same as R_n at $n=0$!]

Remark: this model has been used to predict the spread of covid-19 (often extensions of this model)

Observations: $S_{n+1} - S_n = -\beta S_n I_n \leq 0 \Rightarrow S_n$ is always decreasing

$I_{n+1} - I_n = I_n(\beta S_n - \nu) \Rightarrow I_n$ increases if $S_n > \frac{\nu}{\beta}$ and decreases if $S_n < \frac{\nu}{\beta}$

$\Rightarrow$ once $S_n < \frac{\nu}{\beta}$, both S_n and I_n decrease

Steady states $\begin{cases} \bar{S} = \bar{S} - \beta \bar{S}\bar{I} \\ \bar{I} = \bar{I} + \beta\bar{S}\bar{I} - \nu\bar{I} \end{cases} \Rightarrow \begin{cases} \bar{S}\bar{I} = 0 \\ \bar{I}(\beta\bar{S} - \nu) = 0 \end{cases} \Rightarrow (\bar{S}, 0)$ are steady states for all $\bar{S}$ $(0 \leq \bar{S} \leq N)$

stability $J(S,I) = \begin{pmatrix} 1-\beta I & -\beta S \\ \beta I & 1-\nu+\beta S \end{pmatrix} \Rightarrow J(\bar{S},0) = \begin{pmatrix} 1 & -\beta\bar{S} \\ 0 & 1-\nu+\beta\bar{S} \end{pmatrix}$

Eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1 - \nu + \beta\bar{S}$

$\lambda_2 > 1$ if $\beta\bar{S} > \nu \Rightarrow \bar{S} > \frac{\nu}{\beta}$ $\quad (0 < \nu < 1 \Rightarrow \lambda_2 > 0)$

Two possible situations.

Case 1 $\frac{\nu}{\beta} > N$ $(R_0 < 1)$

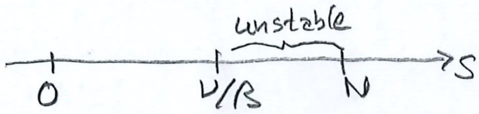
$\bar{S} < N \Rightarrow \bar{S} < \frac{\nu}{\beta} \Rightarrow \lambda_2 < 1$ for all $\bar{S}$, $0 \le \bar{S} \le N$

$\lambda_1 = 1$ is a limit case

$(S_n, I_n) \to (\bar{S}, 0)$, $n \to \infty$. A small change in S_0, I_0 will give a small (but non-zero in general) change in $\bar{S}$.

Both S_n and I_n decrease from the start.

Case 2 $\frac{\nu}{\beta} < N$ $(R_0 > 1)$



If $0 \le \bar{S} < \frac{\nu}{\beta}$, then $\lambda_2 < 1$.

If $\frac{\nu}{\beta} < \bar{S} \le N$, then $\lambda_2 > 1$ and these steady states are unstable.

If (S_0, I_0) has $S_0 < \frac{\nu}{\beta}$, both S_n and I_n decrease from the start, $(S_n, I_n) \to (\bar{S}, 0)$, a small change in (S_0, I_0) gives a small change in $\bar{S}$.

If (S_0, I_0) has $S_0 > \frac{\nu}{\beta}$ (and $I_0 > 0$), I_n initially increases until $S_n < \frac{\nu}{\beta}$, then both decrease and $(S_n, I_n) \to (\bar{S}, 0)$ with $\bar{S} < \frac{\nu}{\beta}$.

These results are essentially the same as for the time continuous SIR model.

See Maple plots with $N = 100$ and $\nu = 0.5$.

For case 1, $\beta = \frac{1}{250}$ $(\Rightarrow R_0 = 0.8)$ is used, and $(S_0, I_0) = (90, 10)$

$(S_n, I_n) \to (\bar{S}, 0)$ with $\bar{S} \approx 72 \Rightarrow R_\infty - R_{(n=0)} \approx 28$ have been infected

For case 2, $\beta = \frac{1}{100}$ $(\Rightarrow R_0 = 2)$

$(S_0, I_0) = (90, 10) \Rightarrow S_0 > \frac{\nu}{\beta} = 50 \Rightarrow I_n$ increases initially

$(S_n, I_n) \to (\bar{S}, 0)$ with $\bar{S} \approx 14 \Rightarrow R_\infty - R_{(n=0)} \approx 86$ were infected

$(S_0, I_0) = (40, 10) \Rightarrow S_0 < \frac{\nu}{\beta} \Rightarrow$ both S_n and I_n decrease

$(S_n, I_n) \to (\bar{S}, 0)$ with $\bar{S} \approx 23 \Rightarrow R_\infty - R_{(n=0)} \approx 77 - 50 = 27$ were infected

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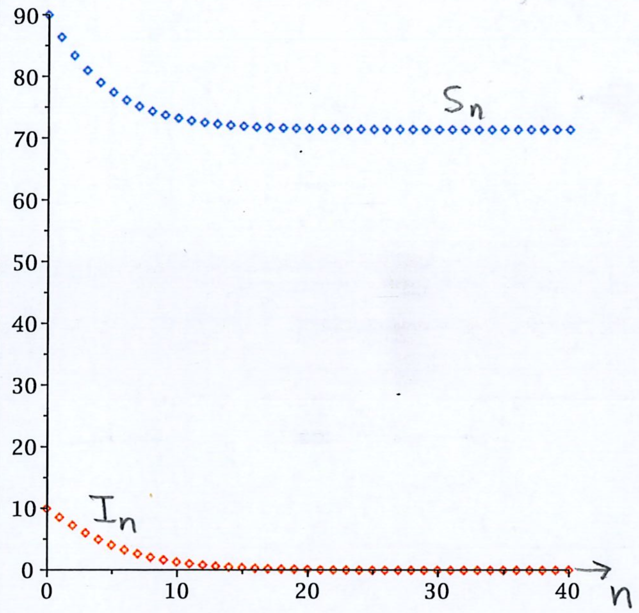
In one exercise, you study a discrete SIRS model. You should find that for $R_0 < 1$, the steady state is at $(N, 0)$. For $R_0 > 1$ the stable steady state is at some $(\bar{S}, \bar{I})$ with $\bar{S} > 0$ and $\bar{I} > 0$. This is again similar to the time cont. SIRS model.

Discrete SIR model, S_n blue, I_n red

$R_0 = 0.8$

$(S_0, I_0) = (90, 10)$

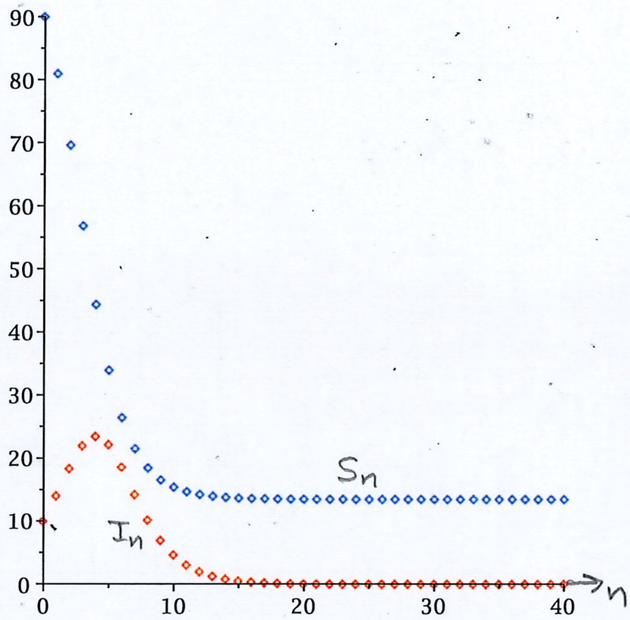
```
> s[0] := 90 : i[0] := 10 :
> for n from 0 to 39 do
    s[n+1] := s[n]·(1 − i[n]/250);
    i[n+1] := i[n]·(0.5 + s[n]/250);
  od:
> with(plots) :
> A := pointplot([seq([n, s[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, i[n]], n = 0..40)], color = red) :
> display(A, B)
```



$R_0 = 2$

$(S_0, I_0) = (90, 10)$

```
> s[0] := 90 : i[0] := 10 :
> for n from 0 to 39 do
    s[n+1] := s[n]·(1 − i[n]/100);
    i[n+1] := i[n]·(0.5 + s[n]/100);
  od:
> A := pointplot([seq([n, s[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, i[n]], n = 0..40)], color = red) :
> display(A, B)
```

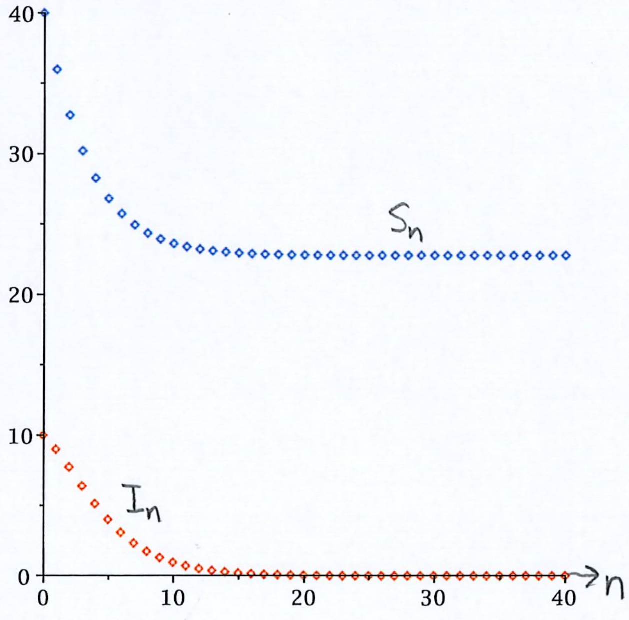



```
> s[0] := 40 : i[0] := 10 :
```

$R_0 = 2$

$(S_0, I_0) = (40, 10)$

```
> for n from 0 to 39 do
    s[n+1] := s[n]·(1 − i[n]/100);
    i[n+1] := i[n]·(0.5 + s[n]/100);
  od:
> A := pointplot([seq([n, s[n]], n = 0..40)], color = blue) :
> B := pointplot([seq([n, i[n]], n = 0..40)], color = red) :
> display(A, B)
```




```
>
```