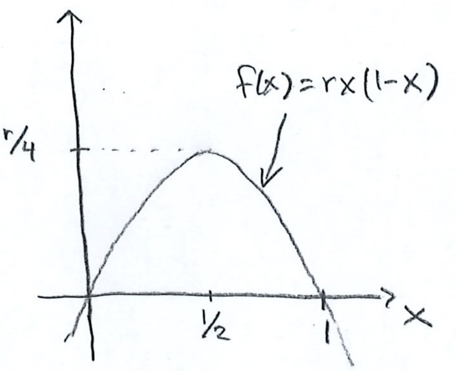
The logistic map (logistic difference eq.)

$x_{n+1} = f(x_n) = r x_n (1-x_n)$ non-linear, similar to some population models (but not exactly one)



If $0 \le x_n \le 1$, then $0 \le x_{n+1} \le \frac{r}{4}$.

Assume $0 \le r \le 4$ for iterations to stay in $[0,1]$ if $x_0 \in [0,1]$.

Steady states $\bar{x} = f(\bar{x}) = r\bar{x}(1-\bar{x}) \Rightarrow \bar{x}_1 = 0$ or $\bar{x}_2 = 1 - \frac{1}{r}$ (if $r > 1$).

$$f'(x) = r(1-2x) \Rightarrow \begin{cases} |f'(0)| = r, & \text{stable if } r<1 \\ |f'(1-\frac{1}{r})| = |2-r|, & \text{stable if } 1<r<3 \end{cases}$$

If $r<1$, $\bar{x}_1 = 0$ is the only steady state and stable $\Rightarrow x_n \to 0, n \to \infty$ for all initial values $x_0 \in [0,1]$.

For $1<r<3$, $x_n \to \bar{x}_2 = 1 - \frac{1}{r}$ as $n \to \infty$ for all x_0 with $0<x_0<1$ ($x_0 = 0$ or $1 \Rightarrow x_1 = x_2 = \ldots = x_n = \ldots = 0, n \ge 2$).

See Wolfram Alpha plots [write "logistic map r=2.5 x_0=0.3" for example]

What happens at $r=3$? $\bar{x}_2$ becomes unstable, $\bar{x}_1$ is unstable $\Rightarrow$ no stable steady states. What happens to x_n as $n \to \infty$?

Check plot for $r=3.1$. It seems that x_n jumps between 2 values, $\tilde{x}_1$ and $\tilde{x}_2$: $f(\tilde{x}_1) = \tilde{x}_2$, $f(\tilde{x}_2) = \tilde{x}_1$. $\Rightarrow (f \circ f)(\tilde{x}_1) = f(f(\tilde{x}_1)) = f(\tilde{x}_2) = \tilde{x}_1$ and $(f \circ f)(\tilde{x}_2) = \tilde{x}_2 \Rightarrow \tilde{x}_1$ and $\tilde{x}_2$ fixed points of $(f \circ f)(x)$.

Study $g(x) = (f \circ f)(x) = f(f(x)) = r f(x)(1-f(x)) = r \cdot rx(1-x)\left(1 - rx(1-x)\right)$

Fixpoints of g? $g(\tilde{x}) = \tilde{x} \Rightarrow r^2 \tilde{x}(1-\tilde{x})(1 - r\tilde{x}(1-\tilde{x})) - \tilde{x} = 0$

Degree 4 but we know $\tilde{x} = \bar{x}_1 = 0$ and $\tilde{x} = \bar{x}_2 = 1 - \frac{1}{r}$ two solutions

$\left(\text{e.g., } g(\bar{x}_2) = f(f(\bar{x}_2)) = f(\bar{x}_2) = \bar{x}_2\right)$

Factorization $\Rightarrow$

$$r^2 \tilde{x}\left(\tilde{x} - \left(1 - \tfrac{1}{r}\right)\right)\underbrace{\left(\tilde{x}^2 - \frac{r+1}{r}\tilde{x} + \frac{r+1}{r^2}\right)}_{(*)} = 0$$

$\circledast = 0 \Rightarrow \tilde{x} = \tilde{x}_{1,2} = \frac{r+1}{2r} \pm \frac{1}{2r}\sqrt{\underbrace{\underbrace{(r+1)^2-4(r+1)}_{=(r+1)(r-3)}}_{<(r+1)^2}}$, real if $r>3$

$\Rightarrow$ also $\tilde{x}_2 > 0$

$\tilde{x}_{1,2}$ stable ?

Calculation gives $|g'(\tilde{x}_1)|<1$ and $|g'(\tilde{x}_2)|<1$ if $3<r<1+\sqrt{6}\approx 3.449$

$\bar{x}_1 = 0$ and $\bar{x}_2 = 1-\frac{1}{r}$ are unstable fixpoints of g if $r>3$.

Means that $f(f(x))$ has 2 stable fixpoints $\tilde{x}_1, \tilde{x}_2$ if $3<r<3.449$ but $f(x)$ has none.

We can verify $f(\tilde{x}_1) = r\tilde{x}_1(1-\tilde{x}_1) = r(-\tilde{x}_1^2+\tilde{x}_1) \underset{\circledast}{=} r(-\frac{r+1}{r}\tilde{x}_1 + \frac{r+1}{r^2} + \tilde{x}_1) = -\tilde{x}_1 + \underbrace{\frac{r+1}{r}}_{=\tilde{x}_1+\tilde{x}_2} = \tilde{x}_2$ OK

$\Rightarrow f(\tilde{x}_2) = f(f(\tilde{x}_1)) = \tilde{x}_1$ OK

We have a stable period-2 oscillation for $3<r<3.449$. Any intitial x_0, $0<x_0<1$, will eventually end up in this oscillation between $\tilde{x}_1$ and $\tilde{x}_2$.

What happens at $r=3.449$? Check plot for $r=3.5$.
$\tilde{x}_{1,2}$ become unstable for g, but $h = g\circ g = f\circ f\circ f\circ f$ gets 4 stable fixpoints $\Rightarrow$ for $3.449<r<3.544$ we have a stable period-4 oscillation

And so on....

r		stable
$r_1 = 1$	$\longrightarrow$	$\bar{x}_2 = 1-\frac{1}{r}$
$r_2 = 3$	$\longrightarrow$	per-2 osc.
$r_3 \approx 3.449$	$\longrightarrow$	per-4 osc.
$r_4 \approx 3.544$	$\longrightarrow$	per-8 osc.
$r_5 \approx 3.5644$		$\vdots$
$\vdots$		
$r_\infty \approx 3.569946$		

One finds $\frac{r_n - r_{n-1}}{r_{n+1}-r_n} \rightarrow \delta \approx 4.669...$ as $n\rightarrow\infty$

Period doublings occur after r-intervals of length $\approx \frac{1}{\delta}$ of previous r-interval

At $r=r_\infty$ this stops, for $r>r_\infty$ the system is chaotic, x_n jumps around in an unpredictable manner, small changes in x_0 give completely different sequences. Check $r=3.6$ plot.

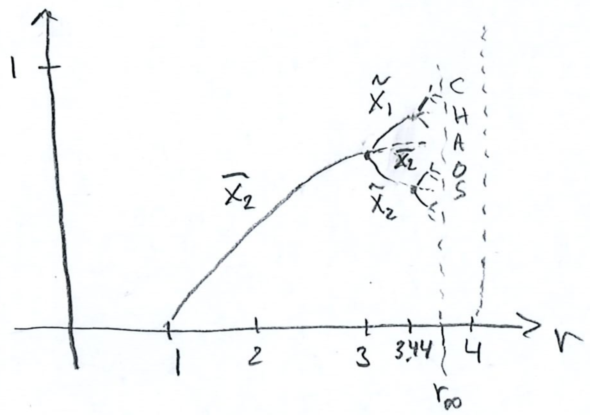
There are some small intervals ("windows") where we have stable oscillations, e.g. near 3.84 we have periods 3, 6, 12, ... ! Try to make plots.

The same number δ appear for all 1D systems with f of shape $\frown$

δ is called Feigenbaum's constant, it is a mathematical constant like π or e (and cannot be expressed in terms of other known constants)

For time continuous systems (ODE's), chaos cannot appear in 1D or 2D ($\sim$Poincaré-Bendixson), but it can in ≥ 3D, e.g. Lorenz meteorological model from 1960's (butterfly effect). Not so easy to define chaos.

Bifurcation diagram [also in Wolfram plot]



— stable

- - - unstable

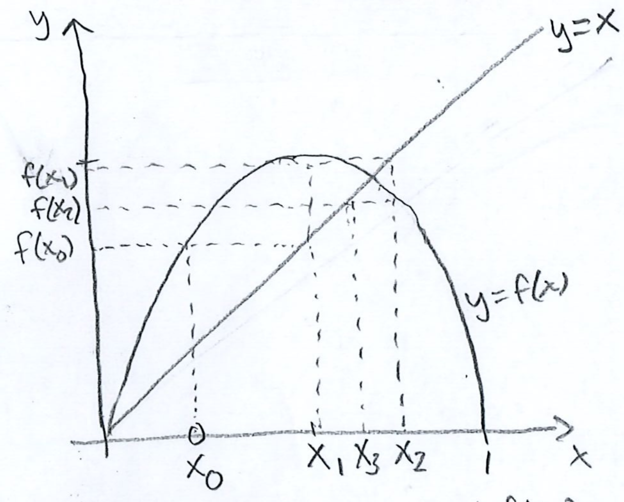
For $r_\infty < r < 4$ there are some narrow windows without chaos.

Cobweb diagrams (a graphical method)

$x_{n+1} = f(x_n)$. Use logistic map $f(x_n) = r x_n (1 - x_n)$ as example

[see also Wolfram plot]

Take $1 < r < 3$



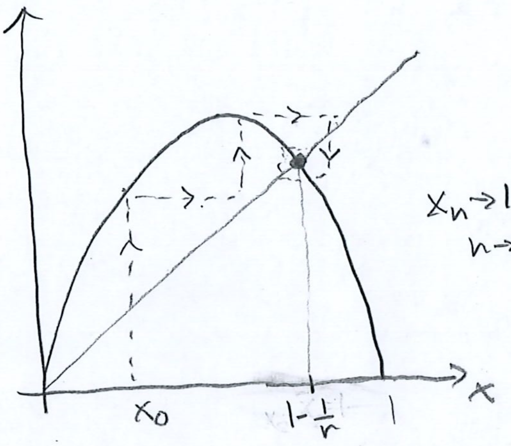
Take x_0, go to curve, mark $f(x_0)$

Go to $y=x$, mark $x_1 = f(x_0)$ on x-axis

continue in this way to see

$x_0, x_1, x_2, x_3, \dots$

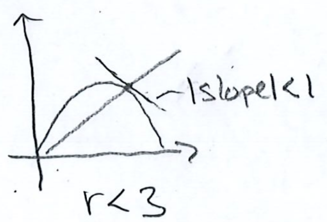
It is sufficient to draw this.

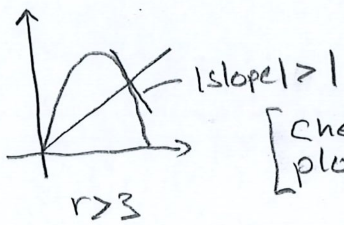


$\updownarrow$ to curve $\quad \leftrightarrow$ to line

Try yourself with different x_0

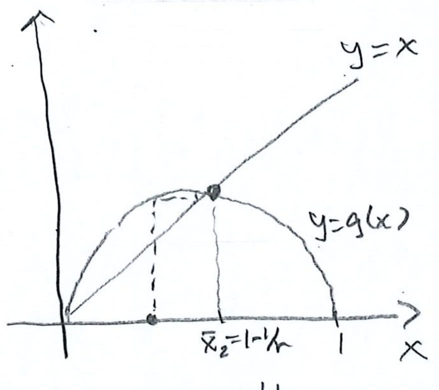
Note: $|f'(1-\frac{1}{r})| < 1$ here. If we increase r to $r > 3$, the fixpoint moves to the right (steeper curve).



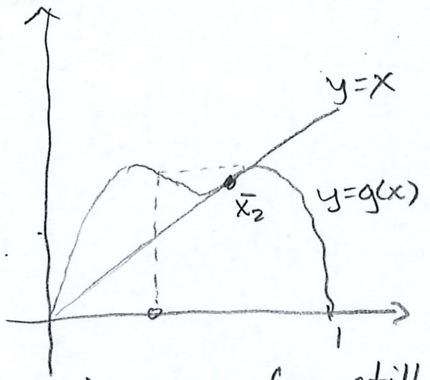


[Check Wolfram plot for $r > 3$]

Cobweb for $g(x) = f(f(x))$: Degree 4 $\Rightarrow$ more complicated to draw. Sketches:

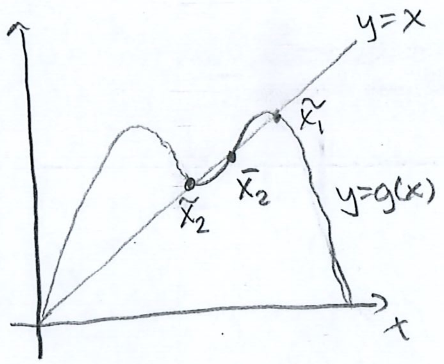


r small



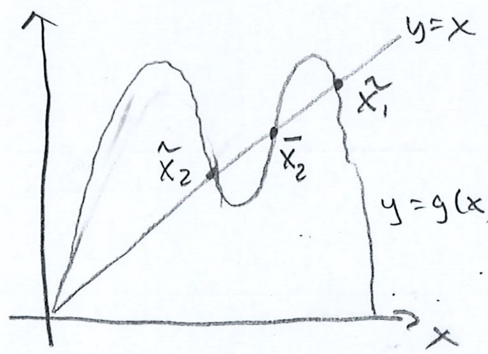
increase of r, still $r<3$

$|g'(\bar{x}_2)|<1$



$3<r<3.449$

$|g'| \begin{cases} <1 \text{ at } \tilde{x}_1, \tilde{x}_2 \text{ (stable)} \\ >1 \text{ at } \bar{x}_2 \text{ (unstable)} \end{cases}$



$r>3.449$

$|g'|>1$ at $\hat{x}_2, \bar{x}_2, \hat{x}_1$

all unstable

Remember to work with exercises on discrete models

15.5

logistic map r=2 x_0=0.2

Extended Keyboard   Upload

Input information:

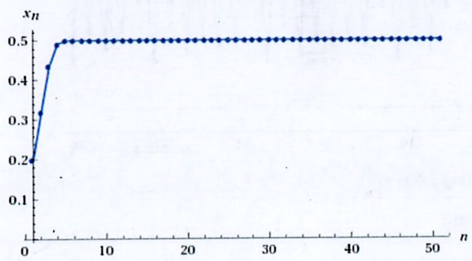
logistic map	
parameter r	2
initial condition x_0	0.2

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

n	0	1	2	3	4
x_n	0.20000	0.32000	0.43520	0.49160	0.49986



Input information:

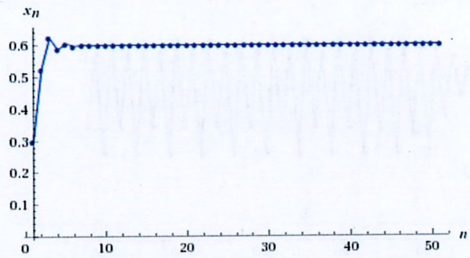
logistic map	
parameter r	2.5
initial condition x_0	0.3

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

n	0	1	2	3	4
x_n	0.30000	0.52500	0.62344	0.58691	0.60612



Limiting behavior:

fixed point

Input information:

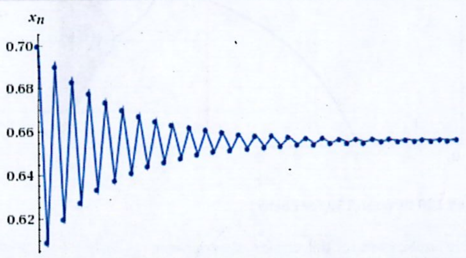
logistic map	
parameter r	2.9
initial condition x_0	0.7

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

n	0	1	2	3	4
x_n	0.70000	0.60900	0.69055	0.61971	0.68344



Limiting behavior:

fixed point

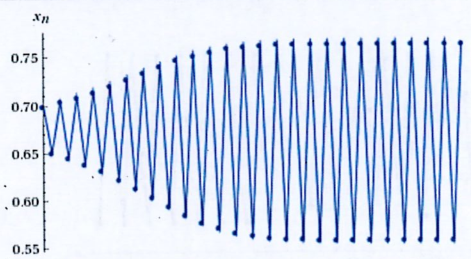
logistic map	
parameter r	3.1
initial condition x_0	0.7

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

n	0	1	2	3	4
x_n	0.70000	0.65100	0.70432	0.64559	0.70929



Limiting behavior:

limit cycle with period 2

Input information:

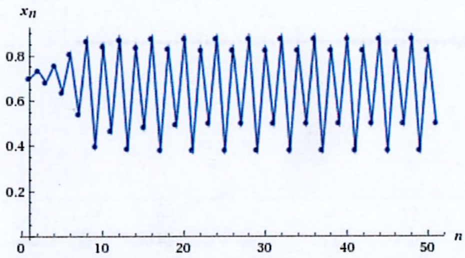
logistic map	
parameter r	3.5
initial condition x_0	0.7

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

n	0	1	2	3	4
x_n	0.70000	0.73500	0.68171	0.75943	0.63943



Limiting behavior:

limit cycle with period 4

Input information:

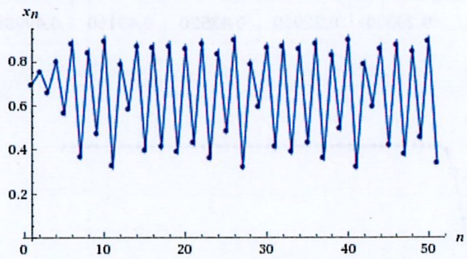
logistic map	
parameter r	3.6
initial condition x_0	0.7

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

n	0	1	2	3	4
x_n	0.70000	0.75600	0.66407	0.80309	0.56929



Limiting behavior:

chaotic

Input information:

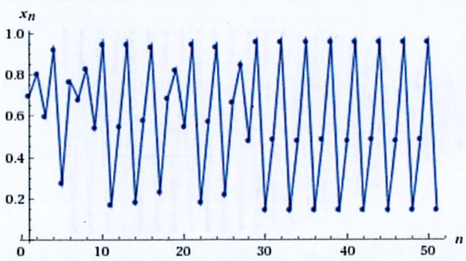
logistic map	
parameter r	3.84
initial condition x_0	0.7

Logistic map:

$x_{n+1} = r\, x_n\,(1 - x_n) \quad (n = 0, 1, 2, \ldots)$

Iterates:

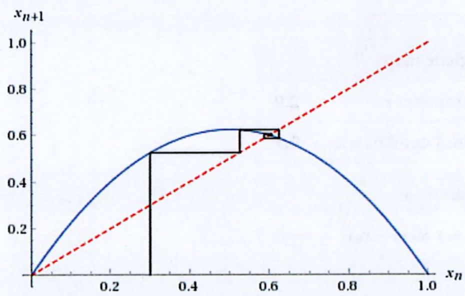
n	0	1	2	3	4
x_n	0.70000	0.80640	0.59950	0.92199	0.27621



Limiting behavior:

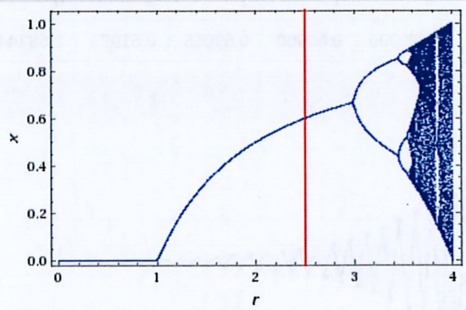
limit cycle with period 3

Cobweb diagram:



(lines successively connect the first 50 iterates and the dashed line $y = x$)

Bifurcation diagram:



(iterates 100 through 150 for each r)

Possible limit cycles for this choice of parameter:

period	iterates	linear stability
1	0	unstable
1	0.6	stable