Population genetics

Let p_n and $q_n = 1-p_n$ be frequencies of alleles A and a in a population at generation n. How do p_n and q_n evolve in time? As $n\to\infty$?

An individual can have genotypes AA, Aa, aa.

We study 3 different mating models, random, equals attract, opposites attract, that lead to different systems of difference equations.

Model 1, random mating (including problem 3.18 of EK)

Assume
1. Mating is random (no preference of, e.g., an AA female to a particular type of male)
2. Same number of progeny for all genotypes (same number of children of AA, Aa, aa)
3. Equal fitness of all progeny
4. No mutations

Let
$$\begin{cases} u_n = \text{frequency of AA at generation } n \\ v_n = \text{— '' — Aa — '' —} \\ w_n = \text{— '' — aa — '' —} \end{cases}$$

$\Rightarrow$ $p_n = u_n + \frac{1}{2}v_n$, $q_n = \frac{1}{2}v_n + w_n$, and $u_n+v_n+w_n = 1$

Mating table.
(shows frequencies of genotypes of progeny)

		Father AA u_n	Father Aa v_n	Father aa w_n
Mother	AA u_n	u_n^2	u_nv_n	u_nw_n
Mother	Aa v_n	u_nv_n	v_n^2	v_nw_n
Mother	aa w_n	u_nw_n	v_nw_n	w_n^2

sum of all nine is 1

Corresponding frequencies of AA, Aa, aa:

$1,0,0$	$\frac{1}{2},\frac{1}{2},0$	$0,1,0$
$\frac{1}{2},\frac{1}{2},0$	$\frac{1}{4},\frac{1}{2},\frac{1}{4}$	$0,\frac{1}{2},\frac{1}{2}$
$0,1,0$	$0,\frac{1}{2},\frac{1}{2}$	$0,0,1$

For the next generation.

Frequency of AA:

$$u_{n+1} = u_n^2 + \tfrac{1}{2}u_nv_n + 0u_nw_n + \tfrac{1}{2}u_nv_n + \tfrac{1}{4}v_n^2 + 0v_nw_n + 0u_nw_n + 0v_nw_n + 0w_n^2 =$$
$$= u_n^2 + u_nv_n + \tfrac{1}{4}v_n^2 = (u_n+\tfrac{1}{2}v_n)^2 = p_n^2$$

Aa: $V_{n+1} = \frac{1}{2}u_nv_n + u_nw_n + \frac{1}{2}u_nv_n + \frac{1}{2}v_n^2 + \frac{1}{2}v_nw_n + u_nw_n + \frac{1}{2}v_nw_n =$

$= u_nv_n + 2u_nw_n + v_nw_n + \frac{1}{2}v_n^2 = 2(u_n + \frac{1}{2}v_n)(\frac{1}{2}v_n + w_n) = 2p_nq_n$

aa: $w_{n+1} = \frac{1}{4}v_n^2 + \frac{1}{2}v_nw_n + \frac{1}{2}v_nw_n + w_n^2 = (\frac{1}{2}v_n + w_n)^2 = q_n^2$

We can write this as a non-linear order-1 system of 3 difference eq.

$$\begin{cases} u_{n+1} = (u_n + \frac{1}{2}v_n)^2 \\ v_{n+1} = 2(u_n + \frac{1}{2}v_n)(\frac{1}{2}v_n + w_n) \\ w_{n+1} = (\frac{1}{2}v_n + w_n)^2 \end{cases}$$

Observe that for p_n and q_n

$p_{n+1} = u_{n+1} + \frac{1}{2}v_{n+1} = p_n^2 + \frac{1}{2}\cdot 2p_nq_n = p_n\underbrace{(p_n + q_n)}_{=1} = p_n \Rightarrow q_{n+1} = 1 - p_{n+1} = 1 - p_n = q_n$

$\Rightarrow$ p_n and q_n remain constant at all times. Hardy-Weinberg law.

[can be deduced directly from

		A	a
A	p_n	p_n^2	p_nq_n
a	q_n	p_nq_n	q_n^2

$p_{n+1} = p_n^2 + \frac{1}{2}p_nq_n + \frac{1}{2}p_nq_n + 0q_n^2 = p_n$]

With $p_n = p$ and $q_n = q = 1-p$ we have

$$\begin{cases} u_{n+1} = p^2 \\ v_{n+1} = 2pq \\ w_{n+1} = q^2 \end{cases}$$ for all $n \geq 0 \Rightarrow u_n, v_n, w_n$ are constant for $n \geq 1$!

$n = 0$?

Tests

	n=0	1	2		n=0	1	2		n=0	1	2		n=0	1	
u_n	1/3	1/4	1/4	...	1/2	1/4	1/4	...	1	1	1	..	0.2	0.2025	0.2025 ...
v_n	1/3	1/2	1/2	...	0	1/2	1/2	...	0	0	0	..	0.5	0.495	0.495 ...
w_n	1/3	1/4	1/4	...	1/2	1/4	1/4	...	0	0	0	..	0.3	0.3025	0.3025 ...

no changes

We see that from n=0 to n=1 it can change.

Note $u_1 - w_1 = (u_0 + \frac{1}{2}v_0)^2 - (\frac{1}{2}v_0 + w_0)^2 = u_0^2 + u_0v_0 - v_0w_0 - w_0^2 = (u_0 - w_0)\underbrace{(u_0 + v_0 + w_0)}_{=1} = u_0 - w_0$

$\Rightarrow$ $u_n - w_n$ is constant from the beginning

For all $n \geq 1$ we have
$$\begin{cases} u_n = (u_0 + \frac{1}{2}v_0)^2 = p^2 \\ v_n = 2(u_0 + \frac{1}{2}v_0)(\frac{1}{2}v_0 + w_0) = 2pq \\ w_n = (\frac{1}{2}v_0 + w_0)^2 = q^2 \end{cases}$$

Model 2, positive assortative mating (including 3.19 of EK)

Modify assumption 1 of model 1 (random mating), to the new

1'. mating only between individuals of the same genotype

Keep 2-4 as in model 1.

Mating table:

		AA	Aa	aa
		u_n	v_n	w_n
AA	u_n	u_n	0	0
Aa	v_n	0	v_n	0
aa	w_n	0	0	w_n

Corresponding frequencies of AA, Aa, aa

1,0,0	–	–
–	$\frac{1}{4},\frac{1}{2},\frac{1}{4}$	–
–	–	0,0,1

For the next generation

$$\begin{cases} u_{n+1} = u_n + \frac{1}{4}v_n \\ v_{n+1} = \frac{1}{2}v_n \\ w_{n+1} = \frac{1}{4}v_n + w_n \end{cases}$$ a linear system

Use $w_n = 1 - u_n - v_n$ and study only $\begin{cases} u_{n+1} = u_n + \frac{1}{4}v_n \\ v_{n+1} = \frac{1}{2}v_n \end{cases} \Leftrightarrow$

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1/4 \\ 0 & 1/2 \end{pmatrix}}_{M} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$. Eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$. Eigenvectors $\bar{s}_1, \bar{s}_2$

$\lambda_1 = 1 \quad (M - I)\bar{s}_1 = \bar{0}: \left(\begin{array}{cc|c} 0 & 1/4 & 0 \\ 0 & -1/2 & 0 \end{array}\right) \Rightarrow \bar{s}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1/2 \quad (M - \frac{1}{2}I)\bar{s}_2 = \bar{0}: \left(\begin{array}{cc|c} 1/2 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{s}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} u_n \\ v_n \end{pmatrix} = c_1 \cdot 1^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \cdot \left(\frac{1}{2}\right)^n \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \cdot \frac{1}{2^n} \\ -2c_2 \cdot \frac{1}{2^n} \end{pmatrix}$$

Initial values ($n=0$): $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -2c_2 \end{pmatrix} \Rightarrow c_2 = -\frac{v_0}{2}, \; c_1 = u_0 + \frac{v_0}{2} \Rightarrow$

$$\begin{cases} u_n = u_0 + \frac{v_0}{2} - \frac{v_0}{2} \cdot \frac{1}{2^n} = u_0 + \frac{v_0}{2}\left(1 - \frac{1}{2^n}\right) \\ v_n = v_0 \cdot \frac{1}{2^n} \\ w_n = 1 - u_n - v_n = w_0 + \frac{v_0}{2}\left(1 - \frac{1}{2^n}\right) \end{cases}$$

Remark: the expression for v_n could have been seen directly

As $n \to \infty$, $\frac{1}{2^n} \to 0$ and

$$\begin{cases} u_n \to u_0 + \frac{v_0}{2} \\ v_n \to 0 \\ w_n \to w_0 + \frac{v_0}{2} \end{cases}$$

Aa dies out, only AA and aa survive

Note that $p_n = u_n + \frac{v_n}{2} = u_0 + \frac{v_0}{2} = p_0$ and $q_n = q_0 = 1 - p_0$ remain constant also here. Also $u_n - w_n = u_0 - w_0$ remains constant.

Tests	n=0	1	2		n=0	1	2	
u_n	1/3	5/12	11/24	... → 1/2	0.2	0.325	0.3875	... → 0.45
v_n	1/3	1/6	1/12	... → 0	0.5	0.25	0.125	... → 0
w_n	1/3	5/12	11/24	... → 1/2	0.3	0.425	0.4875	... → 0.55

Model 3, see next seminar