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Linear ODE's of order 1 and 2

$f(t)$, $g(t)$, $h(t)$ given (one-variable) functions. Find the function $x(t)$ such that

order 1: $\quad x'(t) + f(t)x(t) = g(t) \qquad (1)$

order 2: $\quad x''(t) + f(t)x'(t) + g(t)x(t) = h(t) \qquad (2)$

Both have solution structure $x(t) = x_h(t) + x_p(t)$, where $x_h(t)$ are all homogeneous solutions (the solutions if $g(t) = 0$ in (1) and $h(t) = 0$ in (2)), $x_p(t)$ is one particular solution [true for all linear problems, also systems of linear equations, linear difference equations etc.]

(1) can be solved by multiplying by the integrating factor $e^{F(t)}$, where $F' = f$:

$$\underbrace{e^{F(t)}x'(t) + f(t)e^{F(t)}x(t)}_{=\frac{d}{dt}(e^{F(t)}x(t))} = g(t)e^{F(t)} \quad \Rightarrow \quad e^{F(t)}x(t) = \int g(t)e^{F(t)}dt + C \quad \Rightarrow$$

$$\Rightarrow \quad x(t) = \underbrace{e^{-F(t)} \int g(t)e^{F(t)}dt}_{x_p(t) \ \ (=0 \text{ if } g(t)=0)} + \underbrace{Ce^{-F(t)}}_{x_h(t)}$$

(2) is more difficult for general f, g and h. If $f(t) = a$ and $g(t) = b$ constants, then $x''(t) + ax'(t) + bx(t) = h(t)$ has homogeneous solutions $x_h(t) = c_1e^{r_1t} + c_2e^{r_2t}$ if $r_1 \neq r_2$ ($x_h(t) = (c_1t + c_2)e^{r_1t}$ if $r_1 = r_2$), where $r_{1,2}$ are solutions to $r^2 + ar + b = 0$. If $r_{1,2} = \alpha \pm i\beta$ are complex, this can be written $x_h(t) = e^{\alpha t}(c_3 \cos \beta t + c_4 \sin \beta t)$. $x_p(t)$ is found by some Ansatz (qualified guess).

Separable ODE's of order 1 (can be linear or non-linear)

$f(x)$, $g(t)$ given (one-variable) functions. Find the function $x(t)$ such that

$f(x)x'(t) = g(t)$

Primitive function on $t \quad \Rightarrow \quad F(x(t)) = G(t) + C$

This gives a relation between x and t from which $x(t)$ sometimes can be extracted explicitly.

The solution structure is (in general) not homog. + particular (if non-linear).

Ex. 1D autonomous dynamical system: $\dfrac{dx}{dt} = h(x)$ (no explicit t in the equation) is a separable equation $\dfrac{1}{h(x)} \cdot \dfrac{dx}{dt} = 1 \quad (f(x) = \frac{1}{h(x)},\ g(t) = 1)$

TEST QUESTIONS

1. Solve $x'(t) + 3x(t) = 6$
2. Solve $x'(t) + tx(t) = t$
3. Solve $x''(t) + 6x'(t) + 5x(t) = 0$
4. Solve $x''(t) + 6x'(t) + 9x(t) = 0$
5. Solve $x''(t) + 2x'(t) + 5x(t) = 0$
6. Solve $x(t)^2x'(t) = 2t^5$
7. Solve $x'(t) = x(t)(1 - x(t))$ Hint: $\frac{1}{a(1-a)} = \frac{1}{a} + \frac{1}{1-a}$ (note: one often writes just $x' = x(1-x)$)

ANSWERS NEXT PAGE

ANSWERS

1. $x(t) = 2 + Ce^{-3t}$ (integrating factor e^{3t})
2. $x(t) = 1 + Ce^{-t^2/2}$ (integrating factor $e^{t^2/2}$)
3. $x(t) = C_1 e^{-t} + C_2 e^{-5t}$
4. $x(t) = (C_1 t + C_2)e^{-3t}$
5. $x(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t} = e^{-t}(C_3 \cos 2t + C_4 \sin 2t)$
6. $x(t) = (t^6 + 3C)^{1/3} = (t^6 + K)^{1/3}$ (separable)
7. $(\frac{1}{x} + \frac{1}{1-x})x' = 1 \;\Rightarrow\; \ln|x| - \ln|1-x| = t + C \Rightarrow\; \ln|\frac{x}{1-x}| = t + C \;\Rightarrow\; |\frac{x}{1-x}| = e^{t+C} = e^t e^C \;\Rightarrow$
   $\frac{x}{1-x} = \pm e^C e^t = Ke^t \;\Rightarrow\; xe^{-t} = (1-x)K \;\Rightarrow\; x(K + e^{-t}) = K \;\Rightarrow\; x(t) = \frac{K}{K+e^{-t}}$

The logistic equation

$\frac{dN}{dt} = r(1-\frac{N}{B})N$, r, B constants. Non-linear but separable:

$N' = \frac{r}{B}(B-N)N \Rightarrow \frac{B}{(B-N)N} N' = r \underset{\text{partial fractions}}{\Rightarrow} (\frac{1}{B-N} + \frac{1}{N})N' = r \underset{\text{primitive on } t}{\Rightarrow}$

$-\ln|B-N| + \ln|N| = rt + C \Rightarrow \ln|\frac{N}{B-N}| = rt + C \Rightarrow |\frac{N}{B-N}| = e^{rt+C} = \underbrace{e^C}_{\text{pos. const.}} e^{rt}$

$\Rightarrow \frac{N}{B-N} = Ke^{rt}$ (now $K<0$ also allowed)

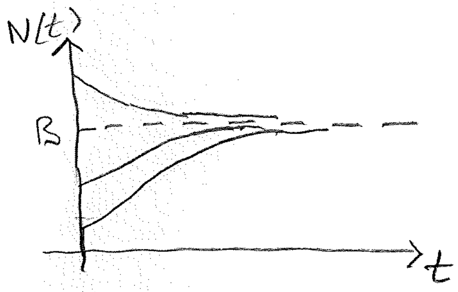
$t=0 \Rightarrow \frac{N_0}{B-N_0} = K \cdot e^0 = K \Rightarrow \frac{N}{B-N} = \frac{N_0}{B-N_0} e^{rt}$. Get N:

$N(B-N_0)e^{-rt} = N_0(B-N) \Rightarrow N(N_0 + (B-N_0)e^{-rt}) = BN_0 \Rightarrow$

$$N(t) = \frac{BN_0}{N_0 + (B-N_0)e^{-rt}}$$ solution to logistic eq. , $N_0 = N(0)$

Note:

1. $N_0 = 0 \Rightarrow N(t) = 0 \quad \forall t > 0$
2. $N_0 = B \Rightarrow N(t) = B \quad \forall t > 0$ — 0 and B are steady states
3. If $N_0 > 0$: $N(t) \to \frac{BN_0}{N_0 + 0} = B$, $t \to \infty$, population approaches B
4. If N_0 small compared to B and t small: $N(t) \approx \frac{BN_0}{Be^{-rt}} = N_0 e^{rt}$
   exponential growth (expected)



solution curves for different N_0

Logistic equation with "fishing term"

$\frac{dN}{dt} = r(1-\frac{N}{B})N - h(N)$, $h(N)$ = harvest, some amount extracted from population

Possible models $h(N) = \begin{cases} E \\ E \cdot N \\ E \cdot N^2 \\ \vdots \end{cases}$ $E > 0$ constant

The equation is still separable

Model $h(N)=EN$ $\Rightarrow N' = rN - \frac{rN^2}{B} - EN = \underbrace{(r-E)}_{=\tilde{r}}N - \frac{rN^2}{B} =$

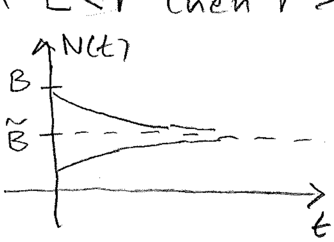
$= \tilde{r}\left(1-\frac{rN}{\tilde{r}B}\right)N = \underbrace{\tilde{r}\left(1-\frac{N}{\tilde{B}}\right)N}_{\text{again a logistic eq.}}$ if $\tilde{B} = \frac{B\tilde{r}}{r}$ and $\tilde{r} = r-E$, new constants

$$\Rightarrow N(t) = \frac{\tilde{B}N_0}{N_0 + (\tilde{B}-N_0)e^{-\tilde{r}t}}$$

Note: if $E>r$ then $\tilde{r}<0$ (and $\tilde{B}<0$) and $e^{-\tilde{r}t} \to \infty, t\to\infty \Rightarrow$

$N(t)\to 0$ and harvest $EN(t)\to 0$, too much fishing (population dies out)

If $E<r$ then $\tilde{r}>0$ and $N(t) \to \tilde{B} = B\underbrace{\left(1-\frac{E}{r}\right)}_{<1} < B,\ t\to\infty$



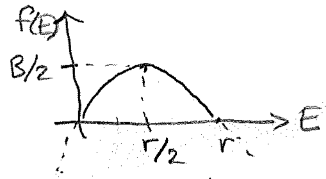
stabilizes at less than carrying capacity

What is maximal harvest at equilibrium? How to choose E?

$N_{eq.} = \tilde{B}$, $h = EN_{eq} = E\tilde{B} = EB\left(1-\frac{E}{r}\right) = BE - \frac{BE^2}{r} = f(E)$. Find max

$f'(E) = B - \frac{2BE}{r} = 0$ if $E = \frac{r}{2}$. Then $h = f\left(\frac{r}{2}\right) =$

$= \frac{rB}{2}\left(1-\frac{1}{2}\right) = \frac{rB}{4}$ and $N_{eq} = \frac{B}{2}$



Maximal harvest reduces N from B to $\frac{B}{2}$ but keeps equilibrium.

Model $h(N) = EN^2$ $\Rightarrow N' = rN - \frac{rN^2}{B} - EN^2 = rN - r\left(\frac{1}{B}+\frac{E}{r}\right)N^2 =$

$= r\left(1 - \frac{1+EB/r}{B}N\right)N = \underbrace{r\left(1-\frac{N}{\hat{B}}\right)N}_{\text{new logistic eq.}}$ if $\hat{B} = \frac{B}{1+EB/r}$

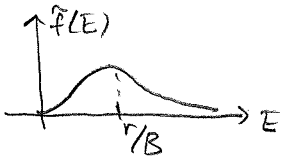
$$\Rightarrow N(t) = \frac{\hat{B}N_0}{N_0 + (\hat{B}-N_0)e^{-rt}}$$ $N(t) \to \hat{B} < B,\ t\to\infty$ (if $N_0>0$)

(population does not die out)

Harvest at equilibrium $h = EN_{eq}^2 = E\hat{B}^2 = \frac{EB^2}{(1+EB/r)^2} = \frac{r^2B^2E}{(r+BE)^2} = \tilde{f}(E)$, max?

$\tilde{f}'(E) = r^2B^2 \frac{1\cdot(r+BE)^2 - 2(r+BE)BE}{(r+BE)^4} = r^2B^2\frac{r+BE-2BE}{(r+BE)^3} = 0$ if $r = BE \Rightarrow E = \frac{r}{B}$

$\Rightarrow h = \tilde{f}\left(\frac{r}{B}\right) = \frac{r^2B^2 \cdot r}{(r+r)^2 \cdot B} = \frac{rB}{4}$ and $N_{eq} = \hat{B} = \frac{B}{2}$



$\Rightarrow$ optimal harvest $\frac{rB}{4}$ and new equilibrium $N_{eq} = \frac{B}{2}$

are the same as in the model $h(N) = EN$!