The problem

$$\begin{cases} u_t(t,x) = D u_{xx}(t,x) & , 0<x<L, t>0 \quad (PDE) \\ u(t,0) = g_1(t) & , t>0 \quad (BC) \\ u(t,L) = g_2(t) & , t>0 \quad (BC) \\ u(0,x) = f(x) & , 0<x<L \quad (IC) \end{cases}$$

is called an <u>initial-boundary value problem</u> (IBVP) for $u(t,x)$ with <u>Dirichlet</u> BC's. With $\begin{cases} u_x(t,0) = g_1(t) \\ u_x(t,L) = g_2(t) \end{cases}$ we have Neumann BC's

Study the IBVP with homogeneous Dirichlet BC's. Take $D=1=L$ for simplicity.

$$\begin{cases} u_t = u_{xx} & , t>0, 0<x<1 & \text{PDE} \quad (1) \\ u(t,0) = 0 & ; t>0 & \text{homog. BC} \quad (2) \\ u(t,1) = 0 & ; t>0 & \\ u(0,x) = f(x) & , 0<x<1 & \text{IC} \quad (3) \end{cases}$$

Look for solutions of the type $u(t,x) = T(t)X(x)$ (good idea?) $\Rightarrow$

$u_t(t,x) = T'(t)X(x)$, $u_x(t,x) = T(t)X'(x)$, $u_{xx}(t,x) = T(t)X''(x)$

(1) $\Rightarrow$ $T'(t)X(x) = T(t)X''(x)$ $\Rightarrow$ (for all t, x)

$$\underbrace{\frac{T'(t)}{T(t)}}_{\text{independent of } x} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{indep. of } t} \quad \text{separation of variables}$$

$\Rightarrow$ both constant (e.g., $\frac{T'(t)}{T(t)} = \frac{X''(1/2)}{X(1/2)} =$ constant for all t)

$$\Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{constant} \Rightarrow \begin{cases} T'(t) - \lambda T(t) = 0 \\ X''(x) - \lambda X(x) = 0 \end{cases} \quad \text{2 standard ODE's}$$

T-eq. $\Rightarrow T(t) = k \cdot e^{\lambda t}$

X-eq: $r^2 - \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}$

$\lambda > 0$ $\quad X(x) = a e^{\sqrt{\lambda}x} + b e^{-\sqrt{\lambda}x}$ (4)

$\lambda = 0$ $\quad X(x) = ax + b$ (5)

$\lambda < 0$ $\quad X(x) = a e^{i\sqrt{|\lambda|}x} + b e^{-i\sqrt{|\lambda|}x} = c\cos\sqrt{|\lambda|}x + d\sin\sqrt{|\lambda|}x$ (6)

Now, check the BC (2)

$$\begin{cases} u(t,0) = T(t)X(0) = 0 \\ u(t,1) = T(t)X(1) = 0 \end{cases} \text{for all } t \Rightarrow \underbrace{T(t)=0 \text{ for all } t}_{\Rightarrow u=0, \text{ not interesting}} \text{ or } \underbrace{X(0)=X(1)=0}_{(7)}$$

Apply (7) to (4), (5) and (6)

$(4) \Rightarrow \begin{cases} X(0) = a+b=0 \\ X(1) = ae^{\sqrt{\lambda}}+be^{-\sqrt{\lambda}}=0 \end{cases} \Rightarrow b=-a \Rightarrow a(\underbrace{e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}}_{\neq 0})=0 \Rightarrow a=0 \Rightarrow b=0$

$\Rightarrow X(x)=0$ for all x (not interesting)

$(5) \Rightarrow \begin{cases} X(0) = b=0 \\ X(1) = a+b=0 \end{cases} \Rightarrow a=b=0 \Rightarrow X(x)=0$ again

$(6) \Rightarrow \begin{cases} X(0) = c=0 \\ X(1) = c\cos\sqrt{|\lambda|}+d\sin\sqrt{|\lambda|}=0 \end{cases} \Rightarrow d\sin\sqrt{|\lambda|}=0 \Rightarrow \underbrace{d=0}_{\Rightarrow X(x)=0 \text{ again}}$ or $\sin\sqrt{|\lambda|}=0$

$\sin\sqrt{|\lambda|}=0 \Rightarrow \underbrace{\sqrt{|\lambda|}}_{>0}=n\pi,\ n=1,2,\ldots \Rightarrow \lambda=-n^2\pi^2 \quad (\lambda<0 \text{ in } (6))$

$\Rightarrow$ gives solutions $X_n(x)=d_n\sin(n\pi x)$, $n=1,2,\ldots$

With $\lambda=-n^2\pi^2$, the corresponding $T(t)$-solutions are $T_n(t)=k_n e^{-n^2\pi^2 t}$

$\Rightarrow u_n(t,x)=T_n(t)X_n(x)=\underbrace{k_n d_n}_{\alpha_n} e^{-n^2\pi^2 t}\sin(n\pi x)$ solve

$\begin{cases} u_t=u_{xx} & (1) \\ u(t,0)=u(t,1)=0 & (2) \end{cases}$ for all $n=1,2,\ldots$

(1) is linear and (2) are homogeneous $\Rightarrow$ linear combinations of $u_n(t,x)$ also solve (1) and (2) ("superposition") $\Rightarrow$.

(*) $u(t,x)=\sum_{n=1}^{\infty}\alpha_n e^{-n^2\pi^2 t}\sin(n\pi x)$ solves (1) and (2) for all coeff. $\alpha_1, \alpha_2, \ldots$

Still left: IC (3) $u(0,x)=f(x)$. $t=0$ in (*) $\Rightarrow$

$u(0,x)=\sum_{n=1}^{\infty}\alpha_n\sin(n\pi x)=f(x)$, which we recognize as the sin-series of f

$\Rightarrow$ choose $\alpha_n=2\int_0^1 f(x)\sin(n\pi x)\,dx$ and we have solved the IBVP (1), (2), (3)!

Ex If $f(x)=3\sin\pi x-4\sin 2\pi x$, we identify $\alpha_1=3$, $\alpha_2=-4$ and $\alpha_n=0$ for $n\geq 3$. The solution to the IBVP is

$$u(t,x)=3e^{-\pi^2 t}\sin(\pi x)-4e^{-4\pi^2 t}\sin(2\pi x)$$

Ex If $f(x)=x$, $\alpha_n=2\int_0^1 x\sin(n\pi x)\,dx=2\left[x\frac{-\cos n\pi x}{n\pi}\right]_0^1+2\int_0^1 1\cdot\frac{\cos n\pi x}{n\pi}\,dx=$

$=-\frac{2}{n\pi}\underbrace{\cos n\pi}_{(-1)^n}+\underbrace{2\left[\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1}_{=0\ (\sin n\pi=0)}=\frac{2}{n\pi}(-1)^{n+1} \Rightarrow$

$$u(t,x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2 t}\sin n\pi x=\frac{2}{\pi}\left(e^{-\pi^2 t}\sin\pi x-\frac{1}{2}e^{-4\pi^2 t}\sin 2\pi x+\frac{1}{3}e^{-9\pi^2 t}\sin 3\pi x-\ldots\right)$$

IBVP's with Neumann boundary conditions

Common in our applications!

$$\begin{cases} u_t = Du_{xx} & , t>0, 0<x<L \quad \text{(PDE)} \\ u_x(t,0)=0 & , t>0 \\ u_x(t,L)=0 & , t>0 \\ u(0,x)=f(x) & , 0<x<L \quad \text{(IC)} \end{cases}$$

$u_x(t,0)=0$, $u_x(t,L)=0$: homogeneous Neumann BC $\to$ means: no flux in our out through boundary

$$u(t,x)=T(t)X(x) \Rightarrow \frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{constant} \Rightarrow T(t)=k\cdot e^{\lambda D t}$$

$$BC \Rightarrow X'(0)=X'(L)=0 \quad (*)$$

Study $X''(x)-\lambda X(x)=0$ as before

$\lambda>0 \Rightarrow X(x)=ae^{\sqrt{\lambda}x}+be^{-\sqrt{\lambda}x} \Rightarrow X'(x)=\sqrt{\lambda}(ae^{\sqrt{\lambda}x}-be^{-\sqrt{\lambda}x})$

$$(*) \Rightarrow \begin{cases} a-b=0 \\ ae^{\sqrt{\lambda}L}-be^{-\sqrt{\lambda}L}=0 \end{cases} \Rightarrow a(\underbrace{e^{\sqrt{\lambda}L}-e^{-\sqrt{\lambda}L}}_{\neq 0})=0 \Rightarrow a=b=0 \Rightarrow X(x)=0 \text{ only}$$

$\lambda=0$ $\quad X(x)=ax+b \Rightarrow X'(x)=a \quad (*) \Rightarrow a=0$ but $X(x)=b$ is a solution

$\lambda<0$ $\quad X(x)=a\cos(\sqrt{|\lambda|}x)+b\sin(\sqrt{|\lambda|}x) \Rightarrow$

$$X'(x)=\sqrt{|\lambda|}\left(-a\sin(\sqrt{|\lambda|}x)+b\cos(\sqrt{|\lambda|}x)\right)$$

$$(*) \Rightarrow \begin{cases} b=0 \\ -a\sin(\sqrt{|\lambda|}L)+b\cos(\sqrt{|\lambda|}L)=0 \end{cases}$$ non-zero solution ($a\neq 0$) if $\sin(\sqrt{|\lambda|}L)=0$

$$\Rightarrow \sqrt{|\lambda|}L=n\pi, n=1,2,\dots \Rightarrow \lambda=-\frac{n^2\pi^2}{L^2}$$

$$\Rightarrow X_n(x)=a_n\cos\frac{n\pi x}{L}$$, $n=0$ from $\lambda=0$ can be included here

Linear PDE, homogeneous BC $\Rightarrow$ linear combinations of solutions to PDE+BC are also solutions $\Rightarrow$

$$u(t,x)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\alpha_n e^{-Dn^2\pi^2t/L^2}\cos\frac{n\pi x}{L}$$ solves PDE + BC for all α_n:s

IC : $u(0,x)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi x}{L}=f(x)$, cos-series of $f(x)$

The IC determines all α_n and we get a unique solution.

$$\alpha_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

Ex $L=\pi$ and $f(x)=x$ (general D) gives

$$\alpha_n=\frac{2}{\pi}\int_0^\pi x\cos nx\,dx \underset{\substack{\uparrow\\ n\neq 0}}{=} \underbrace{\frac{2}{\pi}\left[x\frac{\sin nx}{n}\right]_0^\pi}_{=0}-\frac{2}{\pi}\int_0^\pi 1\cdot\frac{\sin nx}{n}\,dx=\frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^\pi =$$

$$=\frac{2}{\pi n^2}\left((-1)^n-1\right)=\begin{cases}-\frac{4}{\pi n^2}, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}$$

$$n=0:\ \alpha_0=\frac{2}{\pi}\int_0^\pi x\cdot\underbrace{\cos 0}_{=1}\,dx=\frac{1}{\pi}\left[x^2\right]_0^\pi=\pi \qquad \Rightarrow$$

$$u(t,x)=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^\infty\frac{(-1)^n-1}{n^2}e^{-Dn^2t}\cos nx =$$

$$=\frac{\pi}{2}-\frac{4}{\pi}\left(e^{-Dt}\cos x+\frac{1}{9}e^{-9Dt}\cos 3x+\frac{1}{25}e^{-25Dt}\cos 5x+\dots\right)$$

Next time:
inhomogenities, extra terms, 2 space dimensions