Inhomogenities, extra terms

Different tricks can handle extra terms

$$\begin{cases} u_t = u_{xx} & 0<x<1,\ t>0 \\ u(t,0) = A \\ u(t,1) = B \\ u(0,x) = f(x) \end{cases}$$

$\Big\}$ not homog. (but constants)

Put $u(t,x) = v(t,x) + w(x) \Rightarrow u_t = v_t,\ u_x = v_x + w'(x),\ u_{xx} = v_{xx} + w''(x)$

$v(t,0) = u(t,0) - w(0) = A - w(0),\ v(t,1) = u(t,1) - w(1) = B - w(1)$

Can we have $\begin{cases} w''(x) = 0 \\ w(0) = A \\ w(1) = B \end{cases}$? $w(x) = ax + b \Rightarrow \begin{cases} w(0) = b \\ w(1) = a+b \end{cases}$ Take $b = A$, $a = B - A$

$\Rightarrow w(x) = (B-A)x + A$ is OK

Then $v(0,x) = u(0,x) - w(x) = f(x) - (B-A)x - A$

IBVP for $v(t,x)$:

$$\begin{cases} v_t = v_{xx} \\ v(t,0) = 0 \\ v(t,1) = 0 \\ v(0,x) = f(x) - (B-A)x - A = g(x) \end{cases}$$

$\Big\}$ homog.

Solve this for $v(t,x)$.

Then $u(t,x) = v(t,x) + (B-A)x + A$ solves the original IBVP.

Extra terms in the PDE

I. $\begin{cases} u_t = u_{xx} - Au \\ u(t,0) = 0 \\ u(t,1) = 0 \\ u(0,x) = f(x) \end{cases}$

To remove $-Au$, put $u(t,x) = v(t,x)e^{\alpha t} \Leftrightarrow$ $v(t,x) = u(t,x)e^{-\alpha t}$. Then

$u_t = v_t e^{\alpha t} + \alpha v e^{\alpha t},\ u_x = v_x e^{\alpha t},\ u_{xx} = v_{xx} e^{\alpha t}$

substitute in the PDE $\Rightarrow$

$$(v_t + \alpha v)e^{\alpha t} = v_{xx}e^{\alpha t} - Ave^{\alpha t} \Rightarrow v_t + \alpha v = v_{xx} - Av$$

Choose $\alpha = -A \Rightarrow u(t,x) = v(t,x)e^{-At}$ or $v(t,x) = u(t,x)e^{At}$

IBVP for $v(t,x)$

$$\begin{cases} v_t = v_{xx} \\ v(t,0) = u(t,0)e^{At} = 0 \\ v(t,1) = u(t,1)e^{At} = 0 \\ v(0,x) = u(0,x)e^{0} = f(x) \end{cases}$$

Solves this for $v(t,x)$.

Then $u(t,x) = v(t,x)e^{-At}$ solves the original IBVP

II. $\begin{cases} u_t = u_{xx} - Au_x \\ u(t,0)=0 \\ u(t,1)=0 \\ u(0,x)=f(x) \end{cases}$ To remove $-Au_x$, put $u(t,x)=v(t,x)e^{\alpha t+\beta x}$
$\iff v(t,x)=u(t,x)e^{-\alpha t-\beta x}$

Then $u_t=(v_t+\alpha v)e^{\alpha t+\beta x}$, $u_x=(v_x+\beta v)e^{\alpha t+\beta x}$,
$u_{xx}=(v_{xx}+\beta v_x+\beta(v_x+\beta v))e^{\alpha t+\beta x}$ Into the PDE $\Rightarrow$

$$(v_t+\alpha v)e^{\alpha t+\beta x}=(v_{xx}+2\beta v_x+\beta^2 v-Av_x-A\beta v)e^{\alpha t+\beta x} \Rightarrow$$

$$v_t=v_{xx}+\underbrace{(2\beta-A)}_{=0 \text{ if } \beta=A/2}v_x+(\underbrace{\beta^2-A\beta v-\alpha}_{=0 \text{ if } \alpha=\beta^2-A\beta=\frac{A^2}{4}-\frac{A^2}{2}=-\frac{A^2}{4}})v$$

$\Rightarrow$ with $u(t,x)=v(t,x)e^{-A^2t/4+Ax/2}$ or $v(t,x)=u(t,x)e^{A^2t/4-Ax/2}$
we get the IBVP for $v(t,x)$:

$$\begin{cases} v_t=v_{xx} \\ v(t,0)=u(t,0)e^{A^2t/4-0}=0 \\ v(t,1)=u(t,1)e^{A^2t/4-A/2}=0 \\ v(0,x)=u(0,x)e^{0-Ax/2}=f(x)e^{-Ax/2}=g(x) \end{cases}$$

Solve this for $v(t,x)$. Then $u(t,x)=v(t,x)e^{-A^2t/4+Ax/2}$ solves the original IBVP.

Remark: These methods also work for Neumann BC's.

Try to solve some exercises.

Example

Solve for $t>0,\ 0<x<2\pi$

$$\begin{cases} u_t = 3u_{xx} - 2u_x & (1) \\ u(t,0)=0 & \\ u(t,2\pi)=0 & \Big\}(2) \\ u(0,x) = e^{x/3}\sin x & (3) \end{cases}$$

Of type II above.

To remove $-2u_x$, put $u(t,x)=v(t,x)e^{\alpha t+\beta x}$
$\Leftrightarrow v(t,x)=u(t,x)e^{-\alpha t-\beta x}$

$$\Rightarrow u_t = (v_t+\alpha v)e^{\alpha t+\beta x},\quad u_x=(v_x+\beta v)e^{\alpha t+\beta x},\quad u_{xx}=(v_{xx}+\beta v_x+\beta(v_x+\beta v))e^{\alpha t+\beta x}$$

$$(1) \Rightarrow (v_t+\alpha v)e^{\alpha t+\beta x} = \left[3(v_{xx}+2\beta v_x+\beta^2 v)-2(v_x+\beta v)\right]e^{\alpha t+\beta x} \Rightarrow$$

$$v_t+\underbrace{\alpha v}_{\text{put }\alpha=*} = 3v_{xx}+\underbrace{(6\beta-2)}_{\text{put }=0}v_x+\underbrace{(3\beta^2-2\beta)}_{*}v$$

Let $\beta=\frac{1}{3},\ \alpha=3\cdot\frac{1}{9}-\frac{2}{3}=-\frac{1}{3}$.

Then
$$\begin{cases} v_t = 3v_{xx} & (5) \\ v(t,0)=u(t,0)e^{-\alpha t}=0 & \\ v(t,2\pi)=u(t,2\pi)e^{-\alpha t-2\pi\beta}=0 & \Big\}(6) \\ v(0,x)=u(0,x)e^{-\beta x} = e^{x/3}\sin x\cdot e^{-x/3}=\sin x & (7) \end{cases}$$

IBVP for $v(t,x)$

Solve as before, homog. Dirichlet BC (6), $L=2\pi$, $D=3$ $\Rightarrow$

$$v(t,x) = \sum_{n=1}^{\infty}\alpha_n e^{-Dn^2\pi^2t/L^2}\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty}\alpha_n e^{-3n^2t/4}\sin\frac{nx}{2}$$ solves (5) and (6) $\forall\alpha_n$

$$(7) \underset{(t=0)}{\Rightarrow} \sum_{n=1}^{\infty}\alpha_n e^0\sin\frac{nx}{2} = \sin x \quad\Rightarrow \alpha_2=1,\ \alpha_n=0 \text{ if } n\neq 2 \Rightarrow$$

$$v(t,x) = e^{-3t}\sin x \quad\Rightarrow \underline{u(t,x)} = v(t,x)e^{-t/3+x/3} = \underline{e^{-10t/3+x/3}\sin x}$$

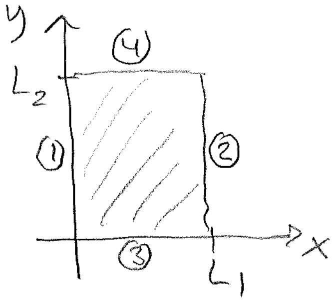
[Note: (3) was chosen to give a simple calculation at the end.
$u(0,x)=f(x) \Rightarrow v(0,x)=f(x)e^{-x/3}$ and $\alpha_n=\frac{2}{2\pi}\int_0^{2\pi}f(x)e^{-x/3}\sin\frac{nx}{2}dx$]

IBVP's in 2 spacedim.

Diffusion/heat equation with BC and IC, $0<x<L_1,\ 0<y<L_2,\ t>0$

$$\begin{cases} u_t = D(u_{xx}+u_{yy}) & (*) \\ u(t,0,y)=0 & (1) \\ u(t,L_1,y)=0 & (2) \\ u(t,x,0)=0 & (3) \\ u(t,x,L_2)=0 & (4) \\ u(0,x,y)=f(x,y) & \text{IC} \end{cases}$$

(1)–(4): homog. Dirichlet BC on the 4 edges.



$$u(t,x,y) = T(t)X(x)Y(y) \text{ in } (*) \Rightarrow \frac{T'(t)}{DT(t)} = \underbrace{\frac{X''(x)}{X(x)}}_{=\lambda} + \underbrace{\frac{Y''(y)}{Y(y)}}_{=\mu} \quad \text{constants}$$

$$T'(t) = D(\lambda+\mu)T(t) \Rightarrow T(t) = k\cdot e^{D(\lambda+\mu)t}$$

$$\left.\begin{aligned} &X''(x)-\lambda X(x)=0 \\ &X(0)=0 \ \text{(from (1))} \\ &X(L_1)=0 \ \text{(from (2))} \end{aligned}\right\} \Rightarrow X_n(x) = \sin\frac{n\pi x}{L_1},\ n=1,2,\dots,\ \lambda = -\frac{n^2\pi^2}{L_1^2}$$

In the same way, $Y_m(y) = \sin\frac{m\pi y}{L_2}$, $m=1,2,\dots$, $\mu = -\frac{m^2\pi^2}{L_2^2}$

$(*)$ linear, BC homogeneous $\Rightarrow$

$$u(t,x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{n,m} e^{-D\left(\frac{n^2}{L_1^2}+\frac{m^2}{L_2^2}\right)\pi^2 t} \sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}$$ solves $(*)$ and BC for all $\alpha_{n,m}$

IC: $u(0,x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{n,m}\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2} = f(x,y)$, this is a 2D sin-series of $f(x,y)$. It can be solved for all (nice) $f(x,y)$ by taking

$$\alpha_{n,m} = \frac{2}{L_1}\cdot\frac{2}{L_2}\int_0^{L_2}\int_0^{L_1} f(x,y)\sin\frac{n\pi x}{L_1}\sin\frac{m\pi y}{L_2}\,dx\,dy$$

With Neumann BC, $\underset{(1)}{u_x(t,0,y)} = \underset{(2)}{u_x(t,L_1,y)} = \underset{(3)}{u_y(t,x,0)} = \underset{(4)}{u_y(t,x,L_2)} = 0$

(no flux through boundaries), one gets 2D cos-series, with terms of the type $\cos\frac{n\pi x}{L_1}\cos\frac{m\pi y}{L_2}$, $n,m=0,1,2,\dots$.