Example, Turing diffusive instability in 1D

Recall
$$\begin{cases} \dfrac{\partial c_1}{\partial t} = R_1(c_1,c_2) + D_1 \dfrac{\partial^2 c_1}{\partial x^2} \\ \dfrac{\partial c_2}{\partial t} = R_2(c_1,c_2) + D_2 \dfrac{\partial^2 c_2}{\partial x^2} \end{cases}$$

With $A = \begin{pmatrix} \frac{\partial R_1}{\partial c_1}(\bar{c}_1,\bar{c}_2) & \frac{\partial R_1}{\partial c_2}(\bar{c}_1,\bar{c}_2) \\ \frac{\partial R_2}{\partial c_1}(\bar{c}_1,\bar{c}_2) & \frac{\partial R_2}{\partial c_2}(\bar{c}_1,\bar{c}_2) \end{pmatrix}$, $(\bar{c}_1,\bar{c}_2)$ a spatially uniform steady state $(R_1(\bar{c}_1,\bar{c}_2) = R_2(\bar{c}_1,\bar{c}_2) = 0)$

we have diffusive instability if
$$\begin{cases} 1.\ \operatorname{Tr} A = a_{11} + a_{22} < 0 \\ 2.\ \det A = a_{11}a_{22} - a_{12}a_{21} > 0 \\ 3.\ a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A} \end{cases}$$

Unstable perturbations, $e^{\sigma t}\cos qx$ with $\sigma > 0$, then appear for q with

$$\left(q^2 - \frac{u}{2}\right)^2 < \frac{u^2}{4} - w, \quad u = \frac{a_{11}D_2 + a_{22}D_1}{D_1 D_2}, \quad w = \frac{\det A}{D_1 D_2}$$

Ex (~problem 11.18 in EK)

A predator-prey model with spatial diffusion is given by

$$\begin{cases} u_t = u + \frac{1}{2}u^2 - uv + \frac{1}{2}u_{xx} & (D_1 = \frac{1}{2}) \\ v_t = uv - v^2 + D v_{xx} & (D_2 = D) \end{cases}$$

For a comparison with Lotka-Volterra, see discussion in EK.

$u(t,x)$ = prey population, $v(t,x)$ = predators

Spatially uniform steady states:

$$\begin{cases} \bar{u}(1 + \frac{1}{2}\bar{u} - \bar{v}) = 0 & (1) \\ \bar{v}(\bar{u} - \bar{v}) = 0 & (2) \Rightarrow \bar{v} = 0 \text{ or } \bar{v} = \bar{u} \end{cases}$$

$\bar{v} = 0$ in (1) $\Rightarrow \bar{u} = 0$ ($\bar{u} = -2 < 0$ not possible)

$\bar{v} = \bar{u}$ in (1) $\Rightarrow \bar{u} = 0$ or $\bar{u} = 2$

$\Rightarrow (\bar{u}_1,\bar{v}_1) = (0,0)$ and $(\bar{u}_2,\bar{v}_2) = (2,2)$

$$A(u,v) = \begin{pmatrix} 1+u-v & -u \\ v & u-2v \end{pmatrix} \Rightarrow$$

$A_1 = A(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\lambda_1 = 1 > 0$, $\lambda_2 = 0$ $\Rightarrow$ unstable without diffusion $\Rightarrow$ diffusive instability not possible

$A_2 = A(2,2) = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}$ $\begin{cases} 1.\ \operatorname{Tr} A_2 = -1 < 0 \\ 2.\ \det A_2 = 2 > 0 \end{cases}$ $\Rightarrow$ (2,2) stable without diffusion

3. becomes $1 \cdot D - 2 \cdot \frac{1}{2} > 2\sqrt{\frac{1}{2} \cdot D \cdot 2} \iff D - 1 > 2\sqrt{D} \Rightarrow$

$$(\sqrt{D} - 1)^2 > 2 \Rightarrow \sqrt{D} > \sqrt{2} + 1 \Rightarrow \underline{D > 3 + 2\sqrt{2} \approx 5.8}$$

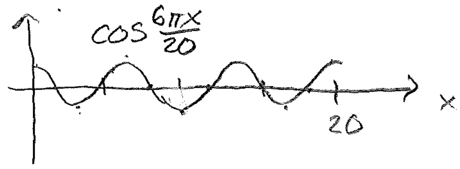
condition on D for diffusive instability

Take $D = 6 \Rightarrow u = \frac{6-1}{6\cdot\frac{1}{2}} = \frac{5}{3}$ and $w = \frac{2}{3} \Rightarrow \frac{u^2}{4} - w = \frac{1}{36}$

unstable perturbations for $-\frac{1}{6} < q^2 - \frac{5}{6} < \frac{1}{6} \Leftrightarrow \frac{2}{3} < q^2 < 1$

With $q = \frac{n\pi}{L}$ and $L = 20$ for example:

$$\frac{2}{3} < \frac{\pi^2}{20^2} n^2 < 1 \Leftrightarrow 27.0 < n^2 < 40.5 \Rightarrow n = 6 \text{ only possibility}$$



intervals with $\cos\frac{6\pi x}{20} > 0$

This is the pattern that can appear due to diffusive instability with our choices of D and L.

Note: if D very near $3+2\sqrt{2}$, $q^2 = \frac{\pi^2 n^2}{L^2}$ must be very near $\frac{5}{6}$ and maybe no integer n satisfies the inequality

This model shows that diffusion can lead to patterns in prey and predator populations. 2D models may be more interesting.

Pattern formation in 2 space dimensions

First we analyze the aggregation of cellular slime molds.

$a(t,x,y)$ = slime mold amoeba

$c(t,x,y)$ = cAMP concentration

Use $\nabla f = (f_x, f_y)$ gradient, $\nabla \cdot (v_1, v_2) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ divergence of vector

$\Rightarrow \nabla \cdot \nabla f = f_{xx} + f_{yy} = \nabla^2 f$ Laplace of f (works also in $\geq$ 3D)

1D	2D (& 3D)
$\frac{\partial a}{\partial t} = -\frac{\partial}{\partial x}(J_{rd} + J_{ch})$	$\frac{\partial a}{\partial t} = -\nabla \cdot (\bar{J}_{rd} + \bar{J}_{ch})$
$J_{rd} = -\mu \frac{\partial a}{\partial x}$	$\bar{J}_{rd} = -\mu \nabla a$ } vectors
$J_{ch} = \chi a \frac{\partial c}{\partial x}$	$\bar{J}_{ch} = \chi a \nabla c$
$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + fa - kc$	$\frac{\partial c}{\partial t} = D \nabla^2 c + fa - kc$

For $\geq$ 2D we have

$$\begin{cases} a_t = \mu \nabla^2 a - \chi \underbrace{\nabla \cdot (a \nabla c)}_{\text{non-lin.}} \underset{\text{product rule}}{=} \mu \nabla^2 a - \chi \underset{\text{scalar prod.}}{\nabla a \cdot \nabla c} - \chi a \nabla^2 c \quad \overset{2D}{\Longrightarrow} \\ c_t = D \nabla^2 c + fa - kc \end{cases}$$

$$\begin{cases} a_t = \mu(a_{xx} + a_{yy}) - \chi(a_x c_x + a_y c_y) - \chi a (c_{xx} + c_{yy}) \\ c_t = D(c_{xx} + c_{yy}) + fa - kc \end{cases}$$

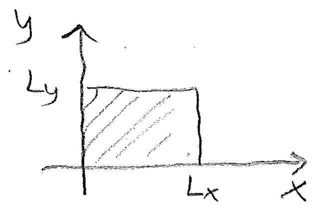
Homogeneous steady state $\bar{a}, \bar{c}$ if all derivatives $0 \Rightarrow f\bar{a} = k\bar{c}$ as in 1D

Perturbations $\begin{cases} a(t,x,y) = \bar{a} + \tilde{a}(t,x,y) \\ c(t,x,y) = \underbrace{\bar{c}}_{\text{constant}} + \underbrace{\tilde{c}(t,x,y)}_{\text{small}} \end{cases}$

Neglecting 2nd-order terms, we find the linearized system

$$\begin{cases} \tilde{a}_t = \mu(\tilde{a}_{xx} + \tilde{a}_{yy}) - \chi \bar{a}(\tilde{c}_{xx} + \tilde{c}_{yy}) \\ \tilde{c}_t = D(\tilde{c}_{xx} + \tilde{c}_{yy}) + f\tilde{a} - k\tilde{c} \end{cases}$$, just as in 1D

With a domain $0 < x < L_x$, $0 < y < L_y$, and no-flux Neumann boundary conditions, look for solutions



$$\begin{cases} \tilde{a}(t,x,y) = A e^{\sigma t} \cos q_1 x \cos q_2 y \\ \tilde{c}(t,x,y) = C e^{\sigma t} \cos q_1 x \cos q_2 y \end{cases}$$

(we could also study, e.g., circular domains but this requires more new studies of solution expressions)

$\Rightarrow$

$$\begin{cases} \sigma\tilde{a} = \mu(-q_1^2\tilde{a} - q_2^2\hat{a}) - \chi\bar{a}(-q_1^2\tilde{c} - q_2^2\hat{c}) \\ \sigma\hat{c} = D(-q_1^2\tilde{c} - q_2^2\hat{c}) + f\hat{a} - k\hat{c} \end{cases}$$

Let $Q^2 = q_1^2 + q_2^2$ and divide by $e^{\sigma t}\cos q_1 x \cos q_2 y \Rightarrow$

$$\begin{cases} \sigma A = -\mu Q^2 A + \chi\bar{a}Q^2 C \\ \sigma C = -DQ^2 C + fA - kC \end{cases}$$ exactly the same as in 1D but q replaced by Q

$\Rightarrow$ solutions with $A \neq 0, C \neq 0$ and $\sigma_1 > 0$ ($\sigma_2 < 0$ always) if

$\underline{\mu(DQ^2 + k) < f\chi\bar{a}}$ condition for aggregation

$q_1 = \dfrac{m\pi}{L_x}$, $q_2 = \dfrac{n\pi}{L_y}$, $m, n = 0, 1, 2, \ldots \Rightarrow Q^2 = \left(\dfrac{m^2}{L_x^2} + \dfrac{n^2}{L_y^2}\right)\pi^2$

If $L_x > L_y$ $\left(\frac{1}{L_x^2} < \frac{1}{L_y^2}\right)$, the first pattern to appear is $m=1, n=0$, i.e. $\cos\frac{\pi x}{L_x}$ ($m=n=0$ not a "pattern"). The second is either $m=0, n=1$, $\cos\frac{\pi y}{L_y}$ if $\frac{1}{L_y^2} < \frac{4}{L_x^2}$ ($L_x < 2L_y$), or $m=2, n=0$, $\cos\frac{2\pi x}{L_x}$ if $L_x > 2L_y$.

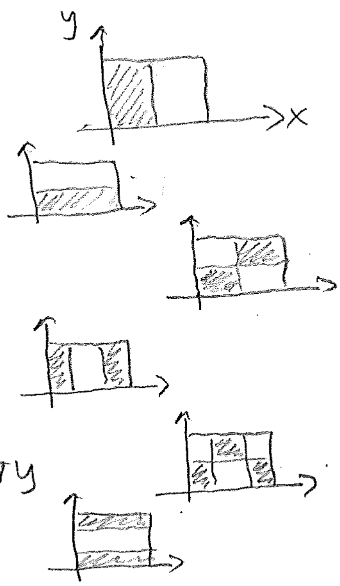
Patterns of amoeba concentrations that can appear satisfy

$$\mu\left(D\pi^2\left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right) + k\right) < f\chi\bar{a}$$

See pictures, page 524 in EK

$\underline{\text{Ex}}$ $L_x = \sqrt{2}$, $L_y = 1$, $Q^2 = \left(\frac{m^2}{2} + n^2\right)\pi^2$, order of appearing:

m, n	$\frac{m^2}{2} + n^2$	function
1, 0	1/2	$\cos\frac{\pi x}{\sqrt{2}}$
0, 1	1	$\cos \pi y$
1, 1	3/2	$\cos\frac{\pi x}{\sqrt{2}} \cos \pi y$
2, 0	2	$\cos\frac{2\pi x}{\sqrt{2}}$
2, 1	3	$\cos\frac{2\pi x}{\sqrt{2}} \cos \pi y$
0, 2	4	$\cos 2\pi y$
3, 0	9/2	$\cos\frac{3\pi x}{\sqrt{2}}$
3, 1	11/2	$\vdots$
2, 2	6	
$\vdots$	$\vdots$	



shows where function positive (where slime molds more lumped together)

Linear combinations of patterns satisfying the aggregation condition give more general patterns for $\tilde{a}$ (and $\tilde{c}$)