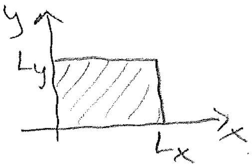
Turing diffusive instability in 2D

Recall that a single diffusion equation for $u(t,x,y)$ in 2D,

$u_t = D\nabla^2 u = D(u_{xx}+u_{yy})$, $t>0$, $0<x<L_x$, $0<y<L_y$,



with no-flux Neumann boundary conditions

$$u_x(t,0,y) = u_x(t,L_x,y) = u_y(t,x,0) = u_y(t,x,L_y) = 0$$

has solutions

$$u(t,x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha_{m,n} e^{-\left(\frac{m^2}{L_x^2}+\frac{n^2}{L_y^2}\right)\pi^2 t} \cos\frac{m\pi x}{L_x}\cos\frac{n\pi y}{L_y}$$

The coefficients $\alpha_{m,n}$ can be determined by an initial condition $u(0,x,y) = f(x,y)$ (2D cos-series).

Consider now a reactor-diffusion system in 2D with $0<x<L_x$, $0<y<L_y$:

$$\begin{cases} \dfrac{\partial c_1}{\partial t} = R_1(c_1,c_2) + D_1\nabla^2 c_1 \\ \dfrac{\partial c_2}{\partial t} = R_2(c_1,c_2) + D_2\nabla^2 c_2 \end{cases}$$

Suppose that $\bar{c}_1, \bar{c}_2$ is a spatially uniform steady state $\begin{cases} R_1(\bar{c}_1,\bar{c}_2)=0 \\ R_2(\bar{c}_1,\bar{c}_2)=0 \end{cases}$

Perturbations $\begin{cases} c_1(t,x,y) = \bar{c}_1 + \tilde{c}_1(t,x,y) \\ c_2(t,x,y) = \bar{c}_2 + \underbrace{\tilde{c}_2(t,x,y)}_{\text{small}} \end{cases}$ gives, as usual, the

linearized system

$$\tilde{C}_t = A\tilde{C} + D\nabla^2\tilde{C} \qquad (*)$$

where $\tilde{C} = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}$, $A = \begin{pmatrix} \frac{\partial R_1}{\partial c_1}(\bar{c}_1,\bar{c}_2) & \frac{\partial R_1}{\partial c_2}(\bar{c}_1,\bar{c}_2) \\ \frac{\partial R_2}{\partial c_1}(\bar{c}_1,\bar{c}_2) & \frac{\partial R_2}{\partial c_2}(\bar{c}_1,\bar{c}_2) \end{pmatrix}$, and $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$

Inspired by the single equation case, we look for solutions

$$\tilde{C} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^{\sigma t}\cos q_1 x \cos q_2 y \qquad \text{(as we did for 2D aggregation)}$$

$$\Rightarrow \tilde{C}_t = \sigma\tilde{C}, \quad \nabla^2\tilde{C} = \frac{\partial^2\tilde{C}}{\partial x^2} + \frac{\partial^2\tilde{C}}{\partial y^2} = -q_1^2\tilde{C} - q_2^2\tilde{C} = -Q^2\tilde{C}, \text{ where } Q^2 = q_1^2 + q_2^2$$

$\circledast \Rightarrow$

$\sigma \tilde{C} = A\tilde{C} - Q^2 D \tilde{C}$, divide by $e^{\sigma t} \cos q_1 x \cos q_2 y \Rightarrow$

$(A - Q^2 D)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \sigma\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$, exactly as in 1D (q^2 replaced by Q^2)

σ eigenvalue and $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ eigenvector of $A - Q^2 D = \begin{pmatrix} a_{11} - Q^2 D_1 & a_{12} \\ a_{21} & a_{22} - Q^2 D_2 \end{pmatrix}$.

With no-flux BC's , $q_1 = \frac{m\pi}{L_x}$, $q_2 = \frac{n\pi}{L_y}$, $Q^2 = \left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right)\pi^2$.

As before , we want

$$\begin{cases} 1.\ \mathrm{Tr} A = a_{11} + a_{22} < 0 \\ 2.\ \det A = a_{11}a_{22} - a_{12}a_{21} > 0 \end{cases}$$ for stability of $\bar{C}_1, \bar{C}_2$ when $D_1 = D_2 = 0$,

and $\det(A - Q^2 D) < 0$ for diffusive instability ($\Rightarrow$ one eigenvalue $\sigma_1 > 0$)

Again, this gives

3. $a_{11} D_2 + a_{22} D_1 > 2\sqrt{D_1 D_2 \det A}$ $(**)$

$\det(A - Q^2 D)$ is minimized (with respect to Q) for $Q^2 = Q_m^2 = \frac{1}{2}\left(\frac{a_{11}}{D_1} + \frac{a_{22}}{D_2}\right)$,

and unstable modes , $e^{\sigma t} \cos \frac{m\pi x}{L_x} \cos \frac{n\pi y}{L_y}$ with $\sigma > 0$, appear if

$$Q_m^2 - \Delta < \left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right)\pi^2 < Q_m^2 + \Delta \quad , \text{where } \Delta = \sqrt{\underbrace{Q_m^4 - \frac{\det A}{D_1 D_2}}_{> 0 \text{ by } (**)}}$$

In section 11.8 of EK , some reactor-diffusion systems that model animal coat patterns are described , see pictures of numerical solutions

These methods for diffusive instabilities work also in 3D.

Effect of geometry on patterns formed [EK problem 11.19]

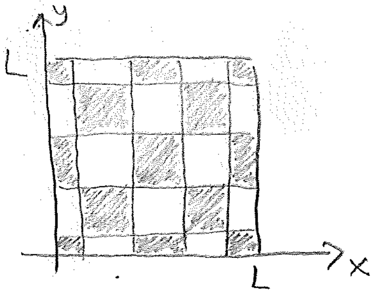
Assume $L_x = L$ and $L_y = \gamma \cdot L$ $\Rightarrow$ $Q^2 = (m^2 + \frac{n^2}{\gamma^2})\frac{\pi^2}{L^2}$

Assume also that L, D_1, D_2 and the matrix A are fixed $\Rightarrow$ Q_m and Δ fixed.

Further, assume Δ very small $\Rightarrow$ small interval for Q^2 so (essentially) only $Q^2 = Q_m^2$ gives diffusive instability and pattern formation. Suppose finally that for $\gamma = 1$, $m = n = 4$ is unstable.

Which m, n will appear when γ increases from $\gamma = 1$ and everything else is fixed? Draw pictures of patterns.

For $\gamma = 1$ ($L_x = L_y = L$), we have $\cos\frac{4\pi x}{L}\cos\frac{4\pi y}{L}$ $(m=n=4)$



shaded areas show where $\cos\frac{4\pi x}{L}\cos\frac{4\pi y}{L} > 0$

$m = n = 4, \gamma = 1 \Rightarrow Q^2 = (16 + \frac{16}{1^2})\frac{\pi^2}{L^2} = 32\frac{\pi^2}{L^2}$

For other γ's, $m^2 + \frac{n^2}{\gamma^2} = 32$ must be kept $\Rightarrow$ $\gamma^2 = \frac{n^2}{32 - m^2}$

m	n	γ^2
4	4	1
4	5	$\frac{25}{16} > 1$ OK
4	6	$\frac{36}{16} > 2$ OK
4	3	$\frac{9}{16} < 1$ not OK (we want $\gamma \geq 1$)
3	4	$\frac{16}{23} < 1$
3	5	$\frac{25}{23} > 1$
3	6	$\frac{36}{23} > 1$
3	7	$\frac{49}{23} > 2$
2	5	$\frac{25}{28} < 1$
2	6	$\frac{36}{28} > 1$
2	7	$\frac{49}{28} > 1$
2	8	$\frac{64}{28} > 2$

m	n	γ^2
1	5	$\frac{25}{31} < 1$
1	6	$\frac{36}{31} > 1$
1	7	$\frac{49}{31} > 1$
1	8	$\frac{64}{31} > 1$
0	5	$\frac{25}{32} < 1$
0	6	$\frac{36}{32} > 1$
0	7	$\frac{49}{32} > 1$
0	8	$\frac{64}{32} = 2$
5	2	$\frac{4}{7} < 1$
5	3	$\frac{9}{7} > 1$
5	4	$\frac{16}{7} > 2$

$m \geq 6$ not possible ($\Rightarrow \gamma^2 < 0$)

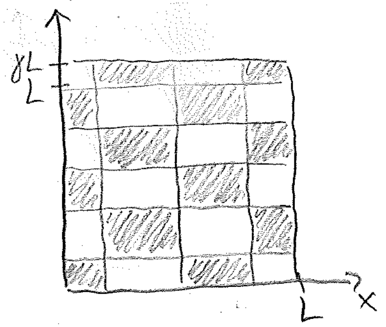
We can now check which patterns that will appear when we increase γ (stretch the domain in the y-direction).

γ	m	n
1	4	4
$\sqrt{\frac{25}{23}} \approx 1.04$	3	5
$\sqrt{\frac{36}{32}} = \sqrt{\frac{9}{8}} \approx 1.06$	0	6
$\sqrt{\frac{36}{31}} \approx 1.08$	1	6
$\sqrt{\frac{9}{7}} \approx 1.13$	2	6
	5	3

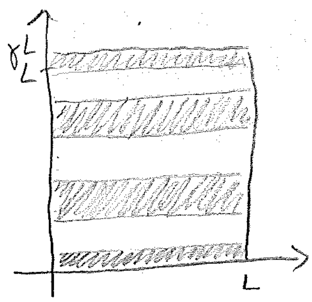
γ	m	n
$\sqrt{\frac{49}{32}} \approx 1.24$	0	7
$\sqrt{\frac{25}{16}} = 1.25$	4	5
$\sqrt{\frac{36}{23}} \approx 1.25$	3	6
$\sqrt{\frac{49}{31}} \approx 1.26$	1	7
⋮		

Pictures

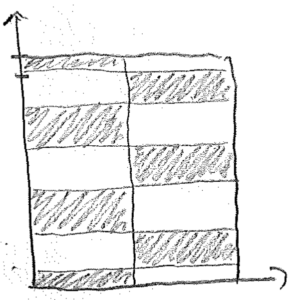
$\gamma = \sqrt{\frac{25}{23}}$



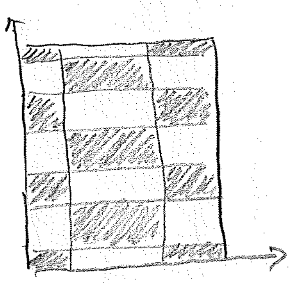
$\gamma = \sqrt{\frac{9}{8}}$



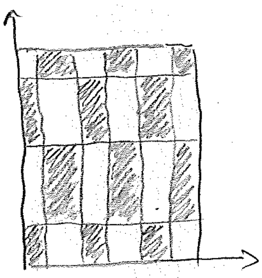
$\gamma = \sqrt{\frac{36}{31}}$



$\gamma = \sqrt{\frac{9}{7}}$



and



With bigger $\Delta > 0$ (longer interval for Q^2), several patterns can appear for a given γ.

On non-rectangular domains, more complex (interesting) and irregular patterns can appear.