Steady states, phase line, stability

$\bar{N}$ is a steady state (equilibrium, critical point) of the auton. eq.

$\frac{dN}{dt} = f(N)$ if $f(\bar{N}) = 0$.

Put $N(t) = \underbrace{\bar{N}}_{\text{constant}} + n(t) \Rightarrow \frac{dN}{dt} = \frac{dn}{dt}$ and for $N(t)$ near $\bar{N}$ ($n(t)$ small)

Taylor $\Rightarrow f(N) = f(\bar{N}+n) = \underbrace{f(\bar{N})}_{=0} + f'(\bar{N})n + \underbrace{\mathcal{O}(n^2)}_{\text{neglect}} \approx f'(\bar{N})n$, linearization of $f(N)$

and $\frac{dn}{dt} \approx f'(\bar{N})n \Rightarrow n(t) \approx k \cdot e^{f'(\bar{N})t} \Rightarrow$

$$n(t) = N(t) - \bar{N} \begin{cases} \to 0 \text{ if } f'(\bar{N}) < 0 & \Rightarrow \bar{N} \text{ stable} \\ \text{grows if } f'(\bar{N}) > 0 & \Rightarrow \bar{N} \text{ unstable} \end{cases}$$

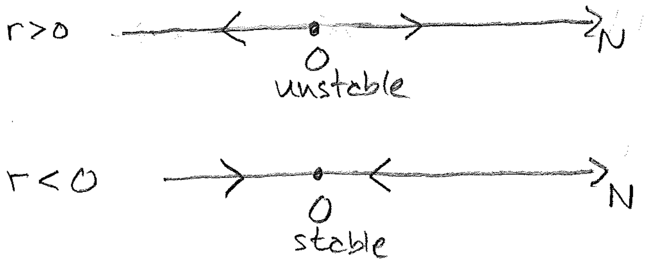
Can also use phase line to see behaviour of solutions:

Ex 1 $\frac{dN}{dt} = f(N) = rN$ (exponential growth)

$f(\bar{N}) = 0 \Rightarrow \bar{N} = 0$ only steady state

$f'(N) = r \Rightarrow f'(0) = r \Rightarrow \bar{N} = 0$ stable if $r < 0$

Phase line



* mark steady states
* mark sign of $f(N)$ with arrows
* shows how solutions move (but not how fast)

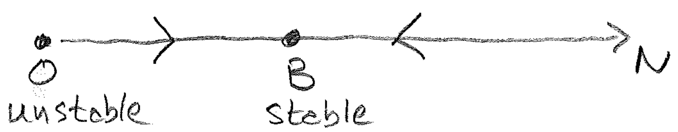
Half-line $N<0$ can be excluded if only $N \geq 0$ possible in application

Ex 2 $\frac{dN}{dt} = f(N) = \frac{r}{B}(B-N)N$ (logistic growth)

$f(\bar{N}) = 0 \Rightarrow \bar{N}_1 = 0, \bar{N}_2 = B$, two steady states

Phase line ($N \geq 0$ assumed)

[Compare with graph of $N(t)$ page 2.3]



$N(t) \to B, t \to \infty$ (if $N(0) > 0$)

Can also check derivative: $f'(N) = \frac{r}{B}(B-2N) \Rightarrow$

$f'(0) = r > 0 \Rightarrow$ unstable

$f'(B) = -r < 0 \Rightarrow$ stable

We see how solutions behave without having an explicit formula.

(Check exercises 1-3!)

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Some linear algebra

A $n \times n$ matrix. $A\bar{v} = \lambda\bar{v}$, $\bar{v} \neq \bar{0}$, λ scalar $\Rightarrow$ $\bar{v}$ eigenvector of A with eigenvalue λ

Step 1. Find all λ by solving $\det(A - \lambda I) = 0$, I = identity matrix. $\det(A - \lambda I)$ is a polynomial of degree n in λ.

Step 2. For each λ, find the corresponding $\bar{v}$ by solving the linear system $(A - \lambda I)\bar{v} = \bar{0}$

$\mathrm{Tr}A$ = trace of A = sum of diagonal elements $= a_{11} + a_{22} + \cdots + a_{nn}$

$$n = 2 \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} =$$

$$\lambda^2 - \underbrace{(a_{11} + a_{22})}_{=\mathrm{Tr}A}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{=\det A} = \lambda^2 - (\mathrm{Tr}A)\lambda + \det A$$

Also, factorizating, $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$

$\Rightarrow \begin{cases} \mathrm{Tr}A = \lambda_1 + \lambda_2 \\ \det A = \lambda_1\lambda_2 \end{cases}$ always. The same is true for $n \times n$ matrices : $\begin{cases} \mathrm{Tr}A = \lambda_1 + \ldots + \lambda_n \\ \det A = \lambda_1 \cdot \ldots \cdot \lambda_n \end{cases}$

(also if some λ_j:s are complex)

$$\det(A - \lambda I) = 0 \Rightarrow \lambda_{1,2} = \frac{\mathrm{Tr}A}{2} \pm \frac{\sqrt{(\mathrm{Tr}A)^2 - 4\det A}}{2}$$

$(\mathrm{Tr}A)^2 - 4\det A = \mathrm{disc}(A)$ = discriminant of A. Sign of $\mathrm{disc}(A)$ determines if $\lambda_{1,2}$ real or complex

Observations

$\lambda_1 > 0,\ \lambda_2 > 0 \Rightarrow \mathrm{Tr}A > 0,\ \det A > 0$
$\lambda_1 < 0,\ \lambda_2 < 0 \Rightarrow \mathrm{Tr}A < 0,\ \det A > 0$
$\lambda_1 > 0,\ \lambda_2 < 0 \Rightarrow \det A < 0$
$\lambda_{1,2} = a \pm ib$ (complex) $\Rightarrow \mathrm{Tr}A = 2a,\ \det A = a^2 + b^2 > 0$

Important for systems of ODE's (of course, some λ_j may be 0)

TEST QUESTIONS (for a real 2×2 matrix A)

1. If $\mathrm{Tr}A = 5$ and $\det A = 4$, find λ_1 and λ_2. Is it OK with the observations?
2. If $\det A = -3$, why are λ_1 and λ_2 real. What are their signs?
3. If $\det A = -3$, can one decide the sign of $\mathrm{Tr}A$?
4. If $\mathrm{Tr}A = -5$ and $\det A = 3$, without calculating λ_1 and λ_2, what are the signs of their real parts?
5. If $\det A = 0$ and $\mathrm{Tr}A = 3$, find λ_1 and λ_2
6. If $\mathrm{Tr}A = 0$ and $\det A = -4$, find λ_1 and λ_2
7. If $\mathrm{Tr}A = 0$ and $\det A = 4$, find λ_1 and λ_2

8*. Find an $A \neq 0$ with $\mathrm{Tr}A = \det A = 0$.

ANSWERS NEXT PAGE

ANSWERS

1. $\lambda_1 = 4$, $\lambda_2 = 1$ (consistent with $\text{Tr}A > 0$ and $\det A > 0$)
2. $\det A = \lambda_1 \lambda_2 < 0$ can only hold if $\lambda_1 > 0$ and $\lambda_2 < 0$ real
3. No, $\text{Tr}A = \lambda_1 + \lambda_2$ can be both positive and negative if $\lambda_1 \lambda_2 < 0$
4. $\text{Tr}A < 0$ and $\det A > 0$ gives either $\lambda_{1,2} < 0$ real or $\lambda_{1,2} = a \pm ib$ with $a < 0$, in both cases $\text{Re}(\lambda_{1,2}) < 0$
5. $\lambda_1 = 3$, $\lambda_2 = 0$
6. $\lambda_1 = 2$, $\lambda_2 = -2$
7. $\lambda_{1,2} = \pm 2i$
8. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

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Linear systems of 2 (order-1) ODE's (constant coefficients)

Find the functions $x(t)$ and $y(t)$ such that $\begin{cases} x'(t) = ax(t) + by(t) \\ y'(t) = cx(t) + dy(t) \end{cases}$ $(a, b, c, d$ constants)

With matrices $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix}$

Eigenvalues and eigenvectors of A: $A\bar{v}_1 = \lambda_1 \bar{v}_1$, $A\bar{v}_2 = \lambda_2 \bar{v}_2$ (assume two independent eigenvectors exist even if $\lambda_1 = \lambda_2$)

Diagonalization of A: $A = TDT^{-1}$ with $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $\bar{v}_1$, $\bar{v}_2$ columns of T.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = TDT^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow T^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = DT^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \text{ Put } T^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = D \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 \tilde{x} \\ \lambda_2 \tilde{y} \end{pmatrix} \Rightarrow \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \tilde{x}\bar{v}_1 + \tilde{y}\bar{v}_2$$

Therefore, all solutions to the system are $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2$

In the case of complex eigenvalues and eigenvectors, $\lambda_{1,2} = \alpha \pm i\beta$, $\bar{v}_{1,2} = \bar{u} \pm i\bar{w}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{(\alpha + i\beta)t} (\bar{u} + i\bar{w}) + c_2 e^{(\alpha - i\beta)t} (\bar{u} - i\bar{w}) =$$

$$= e^{\alpha t} [c_1 (\cos \beta t + i \sin \beta t)(\bar{u} + i\bar{w}) + c_2 (\cos \beta t - i \sin \beta t)(\bar{u} - i\bar{w})] =$$

$$= \underbrace{e^{\alpha t}}_{\text{expon.}} \underbrace{[\overbrace{(c_1 + c_2)}^{K_1}(\bar{u} \cos \beta t - \bar{w} \sin \beta t) + \overbrace{i(c_1 - c_2)}^{K_2}(\bar{u} \sin \beta t + \bar{w} \cos \beta t)]}_{\text{oscillating}}, \quad \text{real expression if } K_1, K_2 \text{ real}$$

If $\lambda_1 = \lambda_2$ and there is only one eigenvector $\bar{v}_1$, the solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda_1 t} [(kc_1 t + c_2)\bar{v}_1 + c_1 \bar{v}_2]$, where $\bar{v}_2$ is an arbitrary vector not parallel to $\bar{v}_1$, and k is determined from $(A - \lambda_1 I)\bar{v}_2 = k\bar{v}_1$ ($(A - \lambda_1 I)\bar{v}_2$ will always be parallel to $\bar{v}_1$). Verification of this solution : $\frac{d}{dt}[e^{\lambda_1 t}((kc_1 t + c_2)\bar{v}_1 + c_1 \bar{v}_2)] = \lambda_1 [e^{\lambda_1 t}((kc_1 t + c_2)\bar{v}_1 + c_1 \bar{v}_2)] + e^{\lambda_1 t} kc_1 \bar{v}_1 = e^{\lambda_1 t}[(kc_1 t + c_2)\lambda_1 \bar{v}_1 + c_1 (k\bar{v}_1 + \lambda_1 \bar{v}_2)] = e^{\lambda_1 t}[(kc_1 t + c_2)A\bar{v}_1 + c_1 A\bar{v}_2] = A[e^{\lambda_1 t}((kc_1 t + c_2)\bar{v}_1 + c_1 \bar{v}_2)]$.

$e^{\lambda t} \to 0$ as $t \to \infty$ if Re$\lambda < 0 \Rightarrow$ real part of eigenvalues determine long-time behaviour of solutions. Recall that Re$\lambda_{1,2} < 0 \Leftrightarrow$ Tr$A < 0$, $\det A > 0$.

Systems of 2 ODE's and single order-2 ODE's:

Note that $x'(t) = ax(t) + by(t) \Rightarrow x''(t) = ax'(t) + by'(t) = ax'(t) + b(cx(t) + dy(t)) = ax'(t) + bcx(t) + d(x'(t) - ax(t)) = (a+d)x'(t) + (bc - ad)x(t) \Leftrightarrow x''(t) - (a+d)x'(t) + (ad - bc)x(t) = 0$. The system therefore gives this order-2 ODE with constant coefficients for $x(t)$. After solving this for $x(t)$, $y(t)$ is determined (if $b \neq 0$) from $by(t) = x'(t) - ax(t)$. Conversely, defining $y(t) = x'(t)$, the order-2 ODE $x''(t) + ax'(t) + bx(t) = 0$ can be written as a system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix}$

Example 1

Solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Solve also the system by rewriting it as an order-2 ODE for $x(t)$.

Solution:

Eigenvalues of $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$, and corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

$\Rightarrow$ solutions are $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

Finding the order-2 ODE for $x(t)$:

$x' = 2x + y\,,\ y' = 4x - y \ \Rightarrow\ x'' = 2x' + y' = 2x' + (4x - y) = 2x' + 4x - (x' - 2x) = x' + 6x \ \Rightarrow$
$x'' - x' - 6x = 0$ which has solution $x(t) = C_1 e^{3t} + C_2 e^{-2t}$. Then
$y(t) = x' - 2x = 3C_1 e^{3t} - 2C_2 e^{-2t} - 2(C_1 e^{3t} + C_2 e^{-2t}) = C_1 e^{3t} - 4C_2 e^{-2t}$

Example 2

2. Solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution:

Eigenvalues of $\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ are $\lambda_{1,2} = 2 \pm i$ with corresponding eigenvectors $\begin{pmatrix} 1 \pm i \\ -1 \end{pmatrix}$ (complex eigenvalues and eigenvectors).

Solutions are $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{(2+i)t} \begin{pmatrix} 1+i \\ -1 \end{pmatrix} + C_2 e^{(2-i)t} \begin{pmatrix} 1-i \\ -1 \end{pmatrix} =$

$= e^{2t}[C_1((\cos t + i \sin t)(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + C_2((\cos t - i \sin t)(\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \end{pmatrix})] =$

$= e^{2t}[\underbrace{(C_1 + C_2)}_{=K_1} \begin{pmatrix} \cos t - \sin t \\ -\cos t \end{pmatrix} + \underbrace{i(C_1 - C_2)}_{=K_2} \begin{pmatrix} \cos t + \sin t \\ -\sin t \end{pmatrix}] = e^{2t} \begin{pmatrix} (K_1 + K_2)\cos t + (K_2 - K_1)\sin t \\ -K_1 \cos t - K_2 \sin t \end{pmatrix}$