Phase planes for linear systems

Ex $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

Eigenvalues $\begin{vmatrix} -1-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 5 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -5$

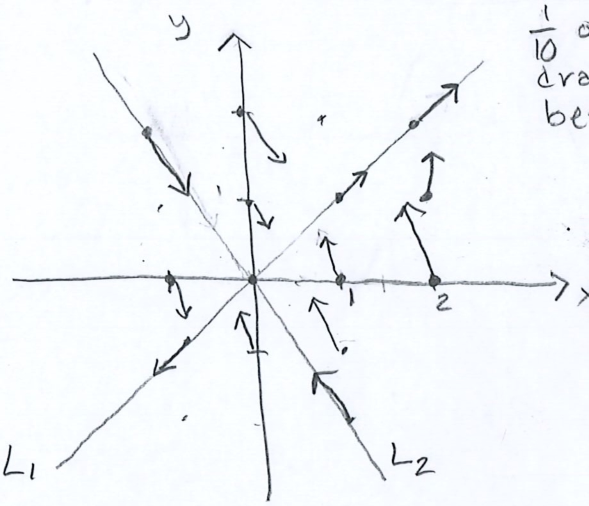
Eigenvectors $\lambda_1 = 1 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -5 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Solutions are $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \underbrace{e^t}_{\to\infty} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \underbrace{e^{-5t}}_{\to 0} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \approx c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for t large

Write $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+2y \\ 4x-3y \end{pmatrix} = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} = \bar{F}(x,y)$

Direction field of $\bar{F}$. At each x, y, draw $\bar{F}(x,y)$ as an arrow

(x, y)	(F_1, F_2)
$(0, 0)$	$(0, 0)$ [unique point with $\bar{F} = \bar{0}$]
$(1, 0)$	$(-1, 4)$
$(2, 0)$	$(-2, 8)$
$(-1, 0)$	$(1, -4)$
$(0, 1)$	$(2, -3)$
$(0, 2)$	$(4, -6)$
$(0, -1)$	$(-2, 3)$
$(1, 1)$	$(1, 1)$
$(1, -1)$	$(-3, 7)$
$(-1, -1)$	$(-1, -1)$
$(2, 1)$	$(0, 5)$
$(2, 2)$	$(2, 2)$
$(-1, 2)$	$(5, -10)$



$\frac{1}{10}$ of length drawn for better picture

L_1 line along $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

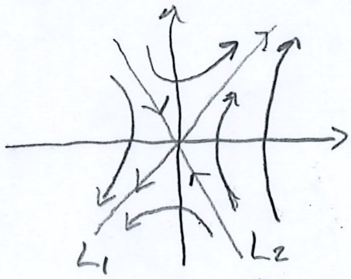
L_2 line along $\bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Better picture: see Wolfram-plot

$\bar{F} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ are tangent vectors to solution curves. Solutions beginning on L_1 will move outwards along L_1, solutions on L_2 move inwards along L_2 towards $(0,0)$. All other curves will approach L_1 and move outwards.

Plot of solution curves:

This is the phase plane for the system



See Wolfram-plot (page 4.5)

Note: only one curve through each point (uniqueness) since x, y determine $\bar{F}$ which determines tangent vectors (e.g., curves cannot cross)

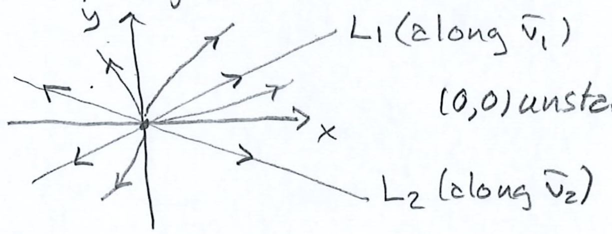
Different cases (phase planes)

Suppose $\lambda_{1,2} \neq 0 \Rightarrow \det A = \lambda_1 \lambda_2 \neq 0 \Rightarrow A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has the unique solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This is for $\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$ to have only $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as steady state.

I. Real eigenvalues (assume 2 indep. eigenvectors if $\lambda_1 = \lambda_2$)

1. $\lambda_1 > 0, \lambda_2 > 0$

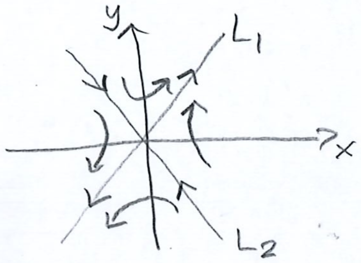
   (solution curves bend towards line with bigger λ_j)



(0,0) unstable

2. $\lambda_1 > 0, \lambda_2 < 0$

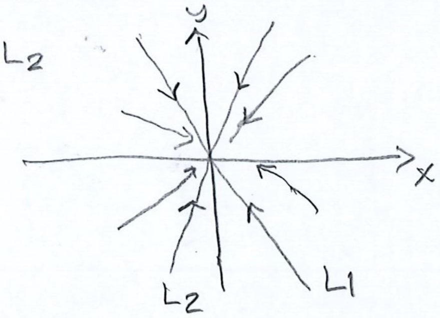
   (our previous example)



saddle point at $(0,0)$, unstable

3. $\lambda_1 < 0, \lambda_2 < 0$

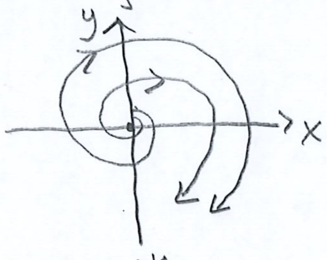
   $(x,y) \to (0,0)$, $t \to \infty$



(0,0) stable

II. Complex eigenvalues $\lambda_{1,2} = a \pm ib$ $(b \neq 0)$

4. $a > 0$

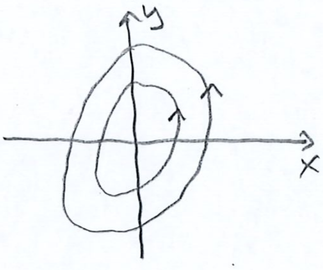


(no point to draw L_1, L_2 since $\bar{v}_1, \bar{v}_2$ complex vectors)

(0,0) unstable

$\|(x,y)\| \to \infty$, $t \to \infty$

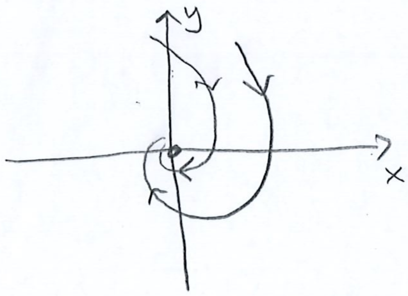
5. $a = 0$



ellipses

(0,0) neutral centre (structurally unstable)

6. $a < 0$

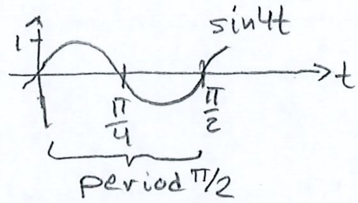


$(x,y) \to (0,0)$, $t \to \infty$

(0,0) stable

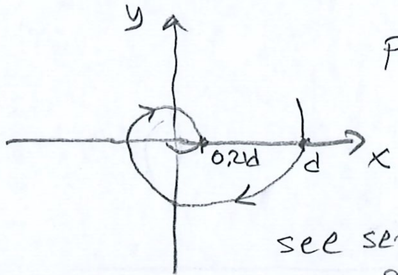
Ex $A = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix}$ has $\lambda_{1,2} = -1 \pm 4i$ and solutions

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{e^{-t}}_{\substack{\text{exponential}\\ \text{decrease}\\ (\text{from } \mathrm{Re}\,\lambda_{1,2})}} \underbrace{\begin{pmatrix} k_1 \cos 4t + k_2 \sin 4t \\ -k_1 \sin 4t + k_2 \cos 4t \end{pmatrix}}_{\substack{\text{oscillating, period } \frac{\pi}{2} \\ (\text{from } \mathrm{Im}\,\lambda_{1,2})}}$$, case 6. above



$$\begin{pmatrix} x(t+\frac{\pi}{2}) \\ y(t+\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot \underbrace{e^{-\pi/2}}_{\approx 0.21}$$

distance to (0,0) reduced to 21% each rotation



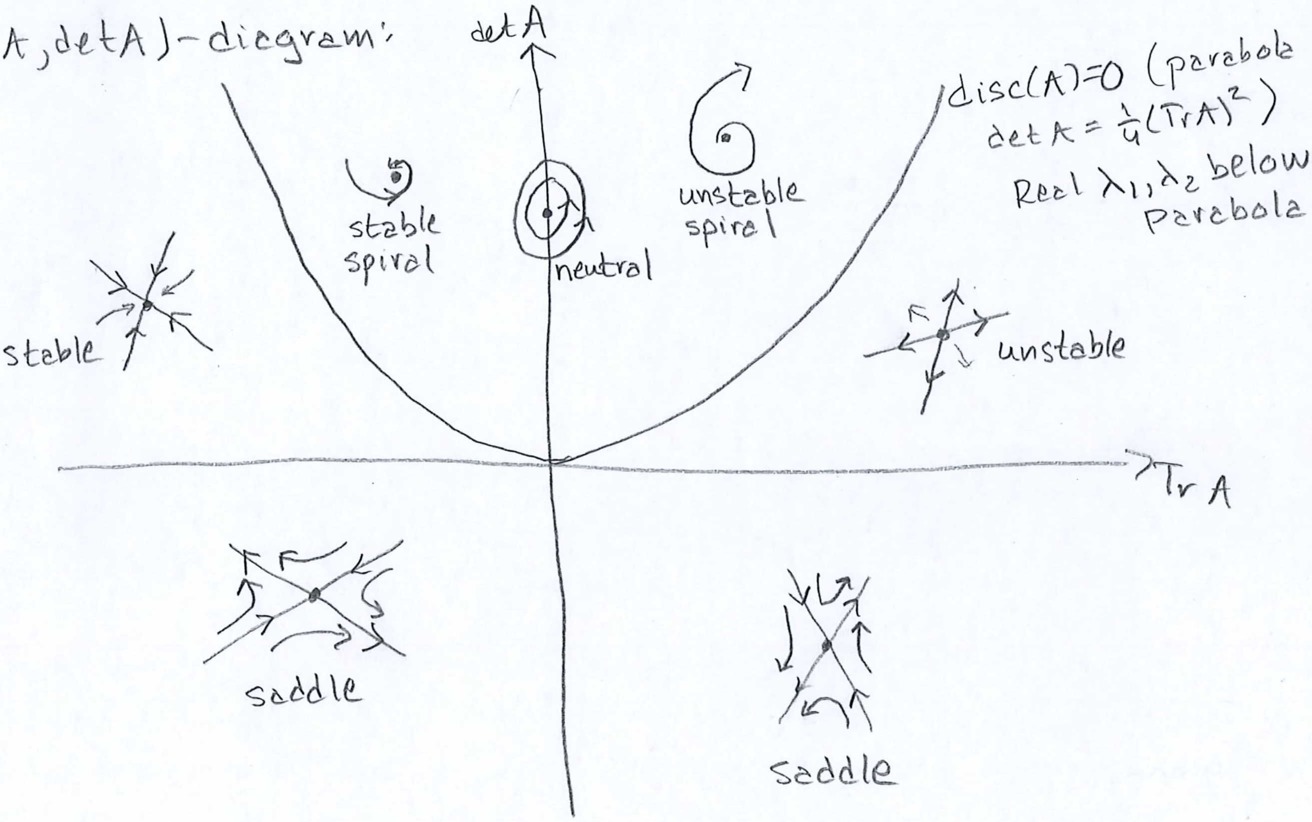
phase plane

see separate plot page 4.6

Recall $\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}A \cdot \lambda + \det A = 0$ (2D)

$$\Rightarrow \lambda_{1,2} = \frac{1}{2}\left(\mathrm{Tr}A \pm \sqrt{\underbrace{(\mathrm{Tr}A)^2 - 4\det A}_{\mathrm{disc}(A)}}\right), \quad \mathrm{Tr}A = \lambda_1 + \lambda_2, \ \det A = \lambda_1 \lambda_2$$

Picture of type of steady state at (0,0) for linear systems in $(\mathrm{Tr}A, \det A)$-diagram:



Matematiska institutionen, Linköpings universitet / Göran Bergqvist

Local stability for non-linear systems, linearization at a steady state

The point $(\bar{x}, \bar{y})$ is a steady state (equilibrium point) of an autonomous (non-linear) system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \quad \text{if } F(\bar{x},\bar{y}) = G(\bar{x},\bar{y}) = 0$$

The type, or character, of a steady state can be studied using Taylor's formula, linearization and the Jacobian matrix.

Let $\begin{cases} x(t) = \bar{x} + \tilde{x}(t) \\ y(t) = \bar{y} + \tilde{y}(t) \end{cases}$ be near $(\bar{x}, \bar{y})$, so $\tilde{x}(t)$ and $\tilde{y}(t)$ are small, and note $\dfrac{dx}{dt} = \dfrac{d\tilde{x}}{dt}, \dfrac{dy}{dt} = \dfrac{d\tilde{y}}{dt}$

Taylor's formula of order 1 and two variables, at $(\bar{x}, \bar{y})$:

$$F(x,y) = F(\bar{x}+\tilde{x}, \bar{y}+\tilde{y}) = \underbrace{F(\bar{x},\bar{y})}_{=0} + F'_x(\bar{x},\bar{y})\tilde{x} + F'_y(\bar{x},\bar{y})\tilde{y} + \underbrace{O(\tilde{x}^2+\tilde{y}^2)}_{\text{neglect}}, \qquad \text{Same for } G(x,y).$$

The original system can near $(\bar{x}, \bar{y})$ be approximated by the linearization

$$\begin{cases} \dfrac{d\tilde{x}}{dt} = F'_x(\bar{x},\bar{y})\tilde{x} + F'_y(\bar{x},\bar{y})\tilde{y} \\ \dfrac{d\tilde{y}}{dt} = G'_x(\bar{x},\bar{y})\tilde{x} + G'_y(\bar{x},\bar{y})\tilde{y} \end{cases} \Leftrightarrow \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{=A} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad \text{where}$$

$$A = \frac{\partial(F,G)}{\partial(x,y)}(\bar{x},\bar{y}) = J(\bar{x},\bar{y}) = \begin{pmatrix} F'_x(\bar{x},\bar{y}) & F'_y(\bar{x},\bar{y}) \\ G'_x(\bar{x},\bar{y}) & G'_y(\bar{x},\bar{y}) \end{pmatrix}$$

is the functional or Jacobian matrix of (F, G) at $(\bar{x}, \bar{y})$ (a constant matrix).

Stability of the steady state $(\bar{x}, \bar{y})$ of the non-linear system and local behaviour of solutions near $(\bar{x}, \bar{y})$ is obtained by studying this linear system with matrix $J(\bar{x}, \bar{y})$.

Linearization can be shown to work if $J(\bar{x}, \bar{y})$ has no eigenvalue with real part 0; this is a sensitive case for linearization since the sign of the real part determines stability. This is called the Hartman-Grobman theorem.

Example

Find the only steady state $(\bar{x}, \bar{y})$ of the system $\begin{cases} \dfrac{dx}{dt} = (x-2)e^y \\ \dfrac{dy}{dt} = y - x^2 \end{cases}$ and calculate $J(\bar{x}, \bar{y})$

Is $(\bar{x}, \bar{y})$ stable?

Solution:

$$\begin{cases} F(\bar{x},\bar{y}) = (\bar{x}-2)e^{\bar{y}} = 0 \\ G(\bar{x},\bar{y}) = \bar{y} - \bar{x}^2 = 0 \end{cases} \Rightarrow (\bar{x},\bar{y}) = (2,4) \quad \text{is the steady state}$$

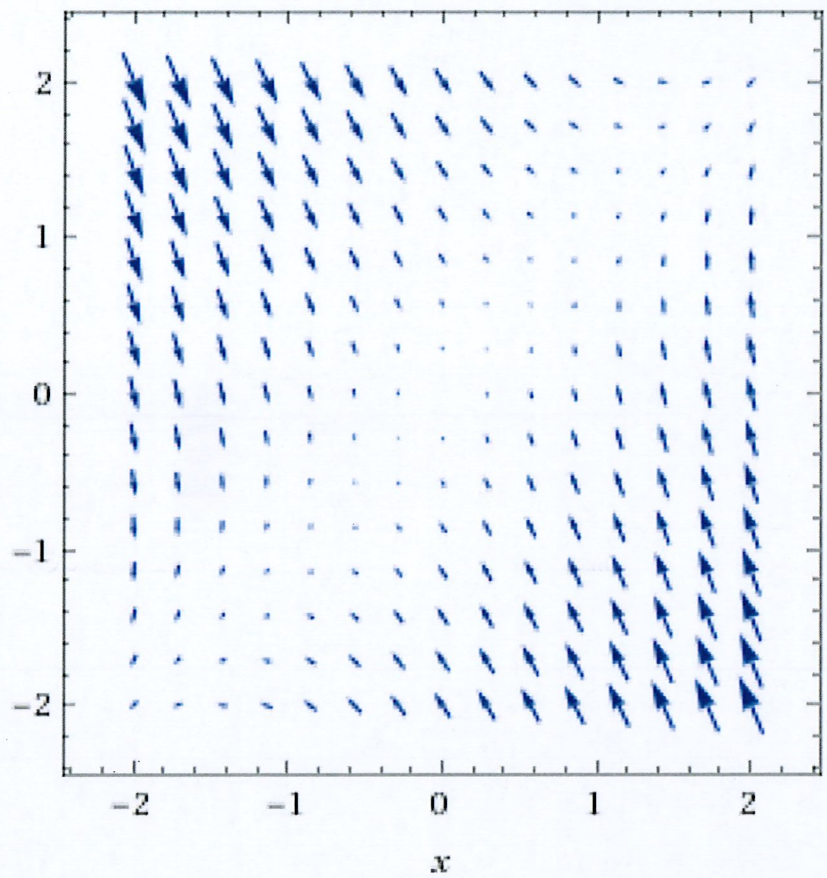
$$J(x,y) = \begin{pmatrix} F'_x(x,y) & F'_y(x,y) \\ G'_x(x,y) & G'_y(x,y) \end{pmatrix} = \begin{pmatrix} e^y & (x-2)e^y \\ -2x & 1 \end{pmatrix} \Rightarrow$$

$J(2,4) = \begin{pmatrix} e^4 & 0 \\ -4 & 1 \end{pmatrix}$ with eigenvalues $\lambda_1 = e^4 > 0$ and $\lambda_2 = 1 > 0 \Rightarrow (2,4)$ is unstable

plot	$-x+2y$	$x = -2$ to 2
	$4x-3y$	$y = -2$ to 2

plot$(-x+2y, 4x-3y)$ $x=-2..2$ $y=-2..2$

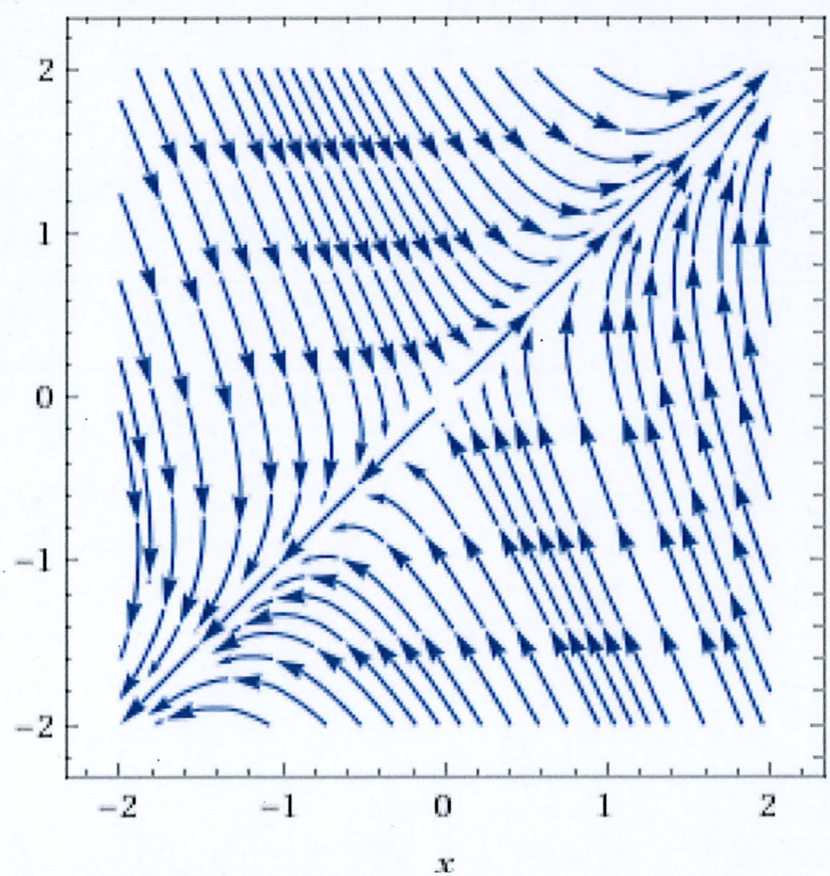
Plot:



Direction field

$$\bar{F} = \begin{pmatrix} -x+2y \\ 4x-3y \end{pmatrix}$$

Integral curves:



Solution curves to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda_1 = 1$ $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -5$ $\bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

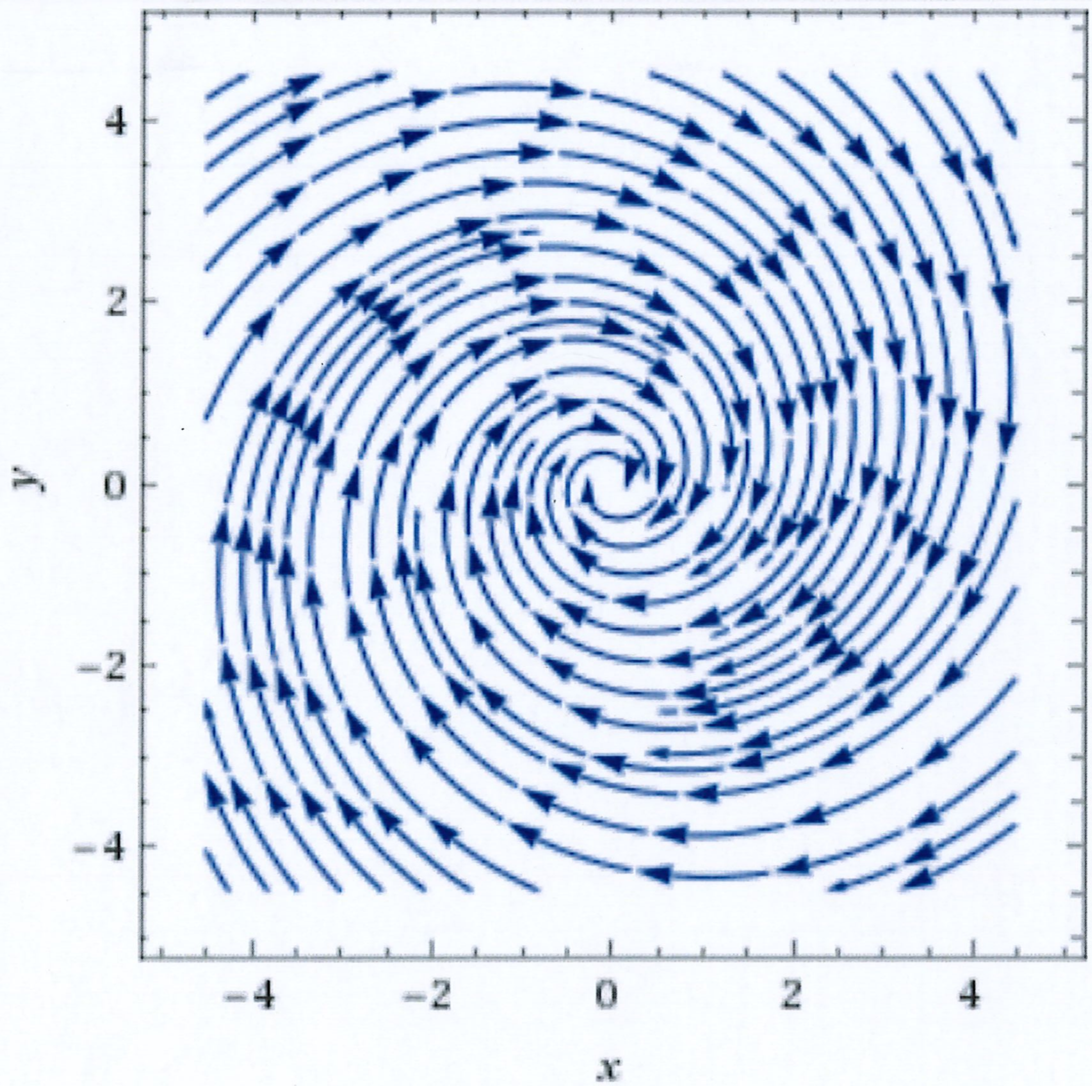
(0,0) unstable (saddle)

Input interpretation:

stream plot	$(-x+4y, -4x-y)$

Solution curves to $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\lambda = -1 \pm 4i$

Plot:



(0,0) stable (spiral)