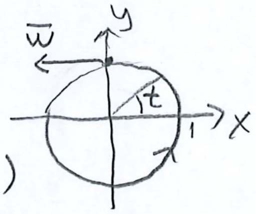
From local to global phase-plane picture

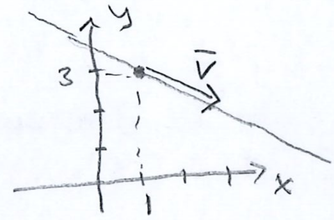
Autonomous dynamical system $\begin{cases} x'(t) = F_1(x(t), y(t)) \\ y'(t) = F_2(x(t), y(t)) \end{cases}$

A solution $(x(t), y(t))$ is a parametrized curve in $\mathbb{R}^2$, t is parameter and $(x'(t), y'(t))$ is a tangent vector (velocity vector) to the curve at the point $(x(t), y(t))$.

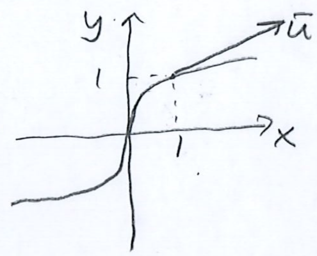
Ex 1. $(x(t), y(t)) = (\cos t, \sin t)$ is the unit circle
$(x'(t), y'(t)) = (-\sin t, \cos t)$ is a tangent vector
$(x'(\frac{\pi}{2}), y'(\frac{\pi}{2})) = \underbrace{(-1, 0)}_{\bar{w}}$ is tangent at $(x(\frac{\pi}{2}), y(\frac{\pi}{2})) = (0,1)$



2. $(x(t), y(t)) = (1+2t, 3-t) = (1,3) + t(2,-1)$ is the line through $(1,3)$ with direction $(2,-1) = \bar{v}$
$(x'(t), y'(t)) = (2,-1) = \bar{v}$ for all t (as expected)



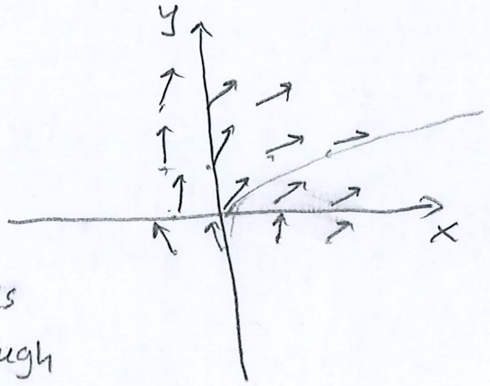
3. $(x(t), y(t)) = (t^3, t)$ is the curve $x = y^3$ or $y = \sqrt[3]{x}$
$(x'(t), y'(t)) = (3t^2, 1)$ is tangent vector
$(x'(1), y'(1)) = (3,1) = \bar{u}$ is tangent at $(x(1), y(1)) = (1,1)$



Direction fields [we have seen for linear systems]

At each (x,y), draw the vector $\bar{F}(x,y) = (F_1(x,y), F_2(x,y))$.
If $(x', y') = \bar{F}$, these vectors are tangent vectors to solution curves.

One can follow the vector field and "see" solution curves.



Uniqueness of solutions guarantees that there is only one curve through each point (x,y).

Nullclines

The <u>x nullcline</u> is the set of points with $F_1(x,y)=0$

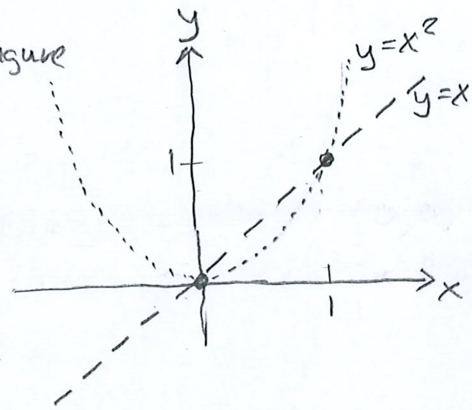
The <u>y nullcline</u> —— " —— $F_2(x,y)=0$

Steady states $\begin{cases} F_1(x,y)=0 \\ F_2(x,y)=0 \end{cases}$ is the intersection of x and y nullclines

<u>Example</u> $\begin{cases} \frac{dx}{dt} = F_1(x,y) = -x+y \\ \frac{dy}{dt} = F_2(x,y) = x^2-y \quad \text{(non-linear)} \end{cases}$

x nullclines $-x+y=0 \Rightarrow y=x$ dashed in figure

y nullclines $x^2-y=0 \Rightarrow y=x^2$ dotted



Steady states: $(0,0)$ and $(1,1)$
(easy to see in figure)

On an x nullcline $\bar{F}=(F_1,F_2)=(0,F_2)$ is vertical
On a y nullcline $\bar{F}=(F_1,0)$ is horizontal

Idea: plot $\bar{F}$ (direction field) on nullclines. Important are the <u>signs</u> of F_2 in $(0,F_2)$ and of F_1 in $(F_1,0)$.

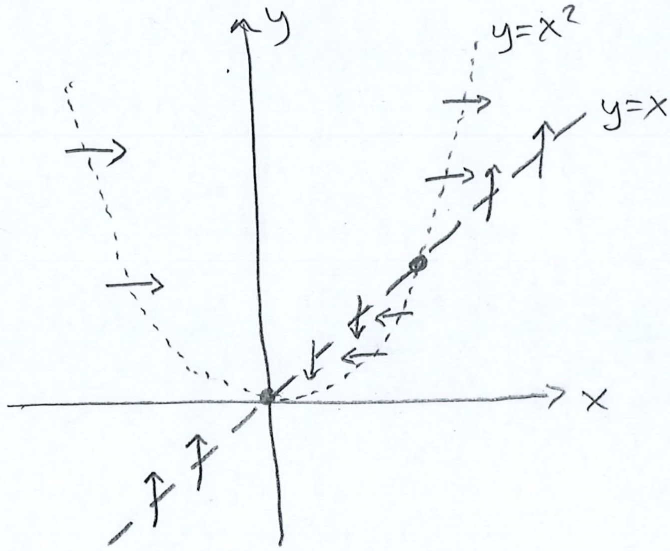
In our example:

$F_2(x,y)=x^2-y \begin{cases} >0 \text{ below } y=x^2 \\ <0 \text{ above } y=x^2 \end{cases}$

Plot $\underbrace{(0,F_2)}_{\text{vertical}}$ along $y=x$

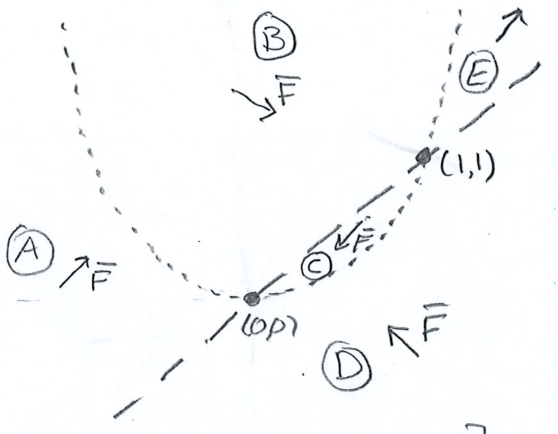
$F_1(x,y)=-x+y \begin{cases} >0 \text{ above } y=x \\ <0 \text{ below } y=x \end{cases}$

Plot $\underbrace{(F_1,0)}_{\text{horizontal}}$ along $y=x^2$



$\bar{F}$ varies in a continuous way, its components F_1 and F_2 can only change sign on nullclines.

We can draw regions and mark the direction of $\bar{F}$ in each



[See also Wolfram plot, page 6.5]

Gives an idea of global behaviour. For example, we can have a solution starting in A, moving into B, where it either moves into E and continue up right, or into C and approaches the steady state $(0,0)$. It could also hit the steady state $(1,1)$ (as $t \to \infty$?) from B, but that seems unlikely since $(1,1)$ looks unstable.

Verify stability properties of the steady states with linearization and Jacobian matrices

$$J(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2x & -1 \end{pmatrix} \Rightarrow$$

$J(0,0) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, eigenvalues $\lambda_1 = \lambda_2 = -1 < 0 \Rightarrow$ stable (as expected)

[in fact, only one eigenvector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$]

$J(1,1) = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$, $\left.\begin{matrix} \lambda_1 = -1+\sqrt{2} > 0 \\ \lambda_2 = -1-\sqrt{2} < 0 \end{matrix}\right\} \Rightarrow$ saddle, unstable (also expected)

(or $\det J(1,1) = -1 < 0$)

The eigenvector to $\lambda_2 = -1-\sqrt{2}$ is $\bar{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$. This is the direction from which one can reach $(1,1)$; but the system is non-linear so it will be along a curve with tangent $\bar{v}_2$ at $(1,1)$. This is difficult to determine in general, the set of points from which solution curves end up at a steady state is the basin of attraction of the steady state.

We will use the above methods to study

* the chemostat
* predator-prey models (Lotka-Volterra)
* populations in competition
* SIR type models for spread of diseases.

Some general remarks for 2D-systems

* solution curves can only intersect at steady states
* solution curve closed $\Rightarrow$ encircles at least one steady state which is not a saddle

Asymptotic behaviour, limitsets:

as $t \to \infty$, solution curves can approach (or already be)

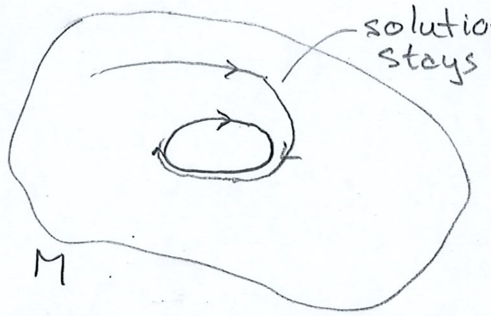
1. a steady state
2. ∞
3. a closed loop  oscillating system (limit cycle if trajectories near it approach it as $t \to \infty$ or $t \to -\infty$, no other closed orbit "near" it)
4. a cycle graph 

$\rightsquigarrow$  heteroclinic

$\rightsquigarrow$  homoclinic

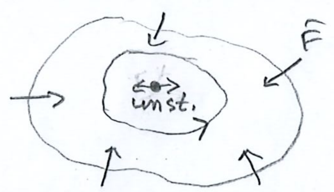
Poincaré-Bendixson theorem (2D) ($F \in C^1$)

If a solution curve is bounded for all $t \geq t_0$ and not approaching a steady state (single point), then it is either a closed orbit or approaches a closed orbit as $t \to \infty$.



[no chaos in 2D!]

Implies (e.g.)



+ flow in only
+ unstable steady state only

$\Rightarrow$ there is a limit cycle inside

Recommended: solve exercises on phase planes for non-linear systems streamplot[{-x+y,x^2-y},{x,-2,4},{y,-2,4}]

Input interpretation:

stream plot	$(-x+y, x^2-y)$	$x = -2$ to 4 $y = -2$ to 4

Plot: