Phase plane for the chemostat

$$\begin{cases} \dfrac{dN}{dt} = \alpha_1 \dfrac{C}{1+C} N - N = F_1(N,C) \\ \dfrac{dC}{dt} = -\dfrac{C}{1+C} N - C + \alpha_2 = F_2(N,C) \end{cases}$$

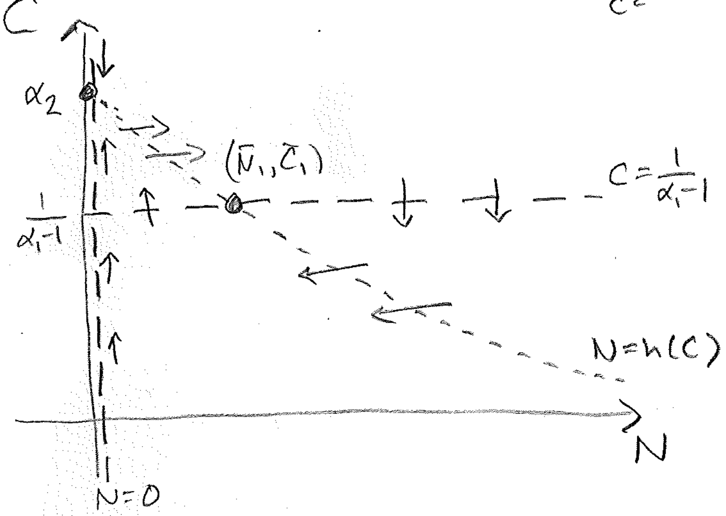
Assume $\alpha_1 > 1, \alpha_2 > \frac{1}{\alpha_1 - 1}$

N nullclines: $N\left(\frac{\alpha_1 C}{1+C} - 1\right) = 0 \Rightarrow N = 0$ (line) or $\frac{\alpha_1 C}{1+C} = 1 \Rightarrow \alpha_1 C = 1 + C \Rightarrow C = \frac{1}{\alpha_1 - 1}$ (line)

C nullclines: $-\frac{CN}{1+C} - C + \alpha_2 = 0 \Rightarrow N = \frac{1+C}{C}(\alpha_2 - C) = \underbrace{\frac{\alpha_2}{C} + \alpha_2 - 1 - C}_{\text{type of curve?}}$

$N = h(C) = \frac{\alpha_2}{C} + \alpha_2 - 1 - C \Rightarrow N \to \infty$ if $C \to 0^+$, $N = 0$ if $C = \alpha_2$

$h'(C) = -\frac{\alpha_2}{C^2} - 1 < 0$, $h''(C) = \frac{2\alpha_2}{C^3} > 0$



Steady states:

$(\bar{N}_1, \bar{C}_1) = \left(\alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right), \frac{1}{\alpha_1 - 1}\right)$

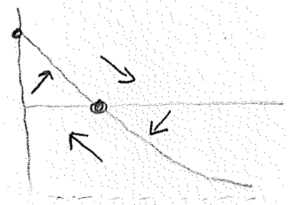
$(\bar{N}_2, \bar{C}_2) = (0, \alpha_2)$

Draw

$\bar{F} = (0, F_2)$ on $N = 0$ and $C = \frac{1}{\alpha_1 - 1}$

with $F_2 \begin{cases} > 0 \text{ under } N = h(C) \\ < 0 \text{ above } N = h(C) \end{cases}$

$\bar{F} = (F_1, 0)$ on $N = h(C)$



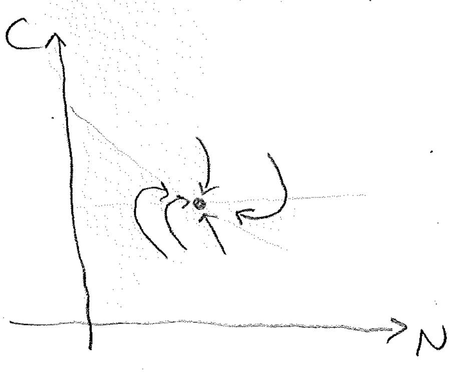
Stability was checked before

$J(\bar{N}_1, \bar{C}_1)$ has $\lambda_1 = -a < 0$, $\lambda_2 = -1 < 0$, stable

$\left(a = \frac{\bar{N}_1}{(1+\bar{C}_1)^2}\right)$

$J(\bar{N}_2, \bar{C}_2)$ has $\lambda_1 = \frac{\alpha_2(\alpha_1 - 1) - 1}{1 + \alpha_2} > 0$, $\lambda_2 = -1 < 0$, saddle unstable

Sketch of phase plane.



See Maple plot with $\alpha_1 = \alpha_2 = 2$ (page 85)

$\left(\Rightarrow \alpha_1 > 1, \alpha_2 > \frac{1}{\alpha_1 - 1}\right)$

$\Rightarrow (\bar{N}_1, \bar{C}_1) = (2, 1)$, $(\bar{N}_2, \bar{C}_2) = (0, 2)$

All initial values $\begin{cases} N(0) > 0 \\ C(0) \geq 0 \end{cases}$ give

solutions $(N(t), C(t)) \to (\bar{N}_1, \bar{C}_1)$

Interpretations of Maple plots

4 figures

1. Direction field and 3 solution curves (blue, brown, and green)
2. $N(t)$ as function of t for the 3 solutions
3. $C(t)$ —— 〃 ——
4. Zoomed in near $(\bar{N}_1, \bar{C}_1)$ for 3 (new) solution curves

I. Blue curve. $N(0)$ and $C(0)$ small. Bacteria density N may decrease but nutrient C is added. After a while N grows and use more C, C begins to decrease. Moves towards $(\bar{N}_1, \bar{C}_1)$

II. Brown curve. $N(0)$ and $C(0)$ at high levels. Bacteria use a lot of nutrient, C decreases and N increases until not enough C, N decrease and we move towards $(\bar{N}_1, \bar{C}_1)$

III. Green curve. $N(0)$ high, $C(0)$ low. Not enough nutrient, N decreases and C then increases. Also moves towards $(\bar{N}_1, \bar{C}_1)$. Shape of final approach depends on initial values.

One can further analyze the approach of solutions to $(\bar{N}_1, \bar{C}_1)$ (problem 5.13 in EK):

$$\frac{dN}{dt} = \alpha_1 \frac{C}{1+C} N - N \quad \text{and} \quad \frac{dC}{dt} = -\frac{C}{1+C} N - C + \alpha_2 \Rightarrow$$

$$\frac{d}{dt}(\underbrace{N + \alpha_1 C}_{g}) = \alpha_1 \frac{C}{1+C} N - N - \alpha_1 \frac{C}{1+C} N - \alpha_1 C + \alpha_1\alpha_2 = -(\underbrace{N + \alpha_1 C}_{g}) + \alpha_1\alpha_2 \Rightarrow$$

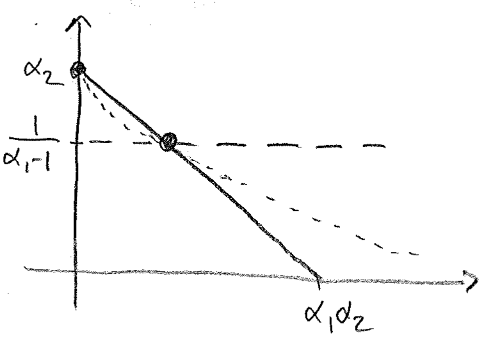
$g' + g = \alpha_1\alpha_2$ simple linear ODE, integrating factor e^t

$$\underbrace{g'e^t + ge^t}_{\frac{d}{dt}(ge^t)} = \alpha_1\alpha_2 e^t \Rightarrow ge^t = \int \alpha_1\alpha_2 e^t dt = \alpha_1\alpha_2 e^t + k \Rightarrow g = \alpha_1\alpha_2 + ke^{-t}$$

$$\Rightarrow N(t) + \alpha_1 C(t) = \alpha_1\alpha_2 + ke^{-t} \quad (*)$$

Let $t \to \infty \Rightarrow N(t) + \alpha_1 C(t) \to \alpha_1\alpha_2 + k \cdot 0 = \alpha_1\alpha_2 \Rightarrow$

Solution curves $(N(t), C(t))$ approach the line $N + \alpha_1 C = \alpha_1\alpha_2$

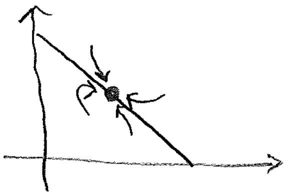


Both $(\bar{N}_1, \bar{C}_1)$ and $(\bar{N}_2, \bar{C}_2)$ are on this line

In the plot, $\alpha_1 = \alpha_2 = 2$, this line is $N + 2C = 4$ or $C = -\frac{N}{2} + 2$

If $(N(0), C(0))$ is on the line ⊛, then $\underbrace{N(0)+\alpha_1 C(0)}_{\alpha_1\alpha_2} = \alpha_1\alpha_2 + k\cdot e^0 \Rightarrow$

$k=0 \Rightarrow N(t)+\alpha_1 C(t) = \alpha_1\alpha_2$ for all t $\Rightarrow$ we stay on the line, the line is a solution curve (in spite of non-linear equations). We move towards $(\bar{N}_1, \bar{C}_1)$ on it. Also means that we cannot cross the line with other solution curves.



In the zoomed Maple plot, the green curve is on the line ($(N(0), C(0)) = (2.1, 0.95)$ is on $N+2C=4$).

Recall that $J(\bar{N}_1, \bar{C}_1)$ has $\lambda_1 = -a$ with $\bar{v}_1 = \begin{pmatrix}\alpha_1\\-1\end{pmatrix}$

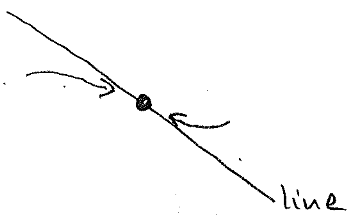
$\lambda_2 = -1$ with $\bar{v}_2 = \begin{pmatrix}\alpha_1 a\\-1\end{pmatrix}$ (assume $a \neq 1$)

$\Rightarrow$ solutions to the linearized system near $(\bar{N}_1, \bar{C}_1)$ have e^{-at} and e^{-t} terms. If $a<1$, $e^{-t} \to 0$ faster than e^{-at}, and for large t solutions are $\approx c_1 e^{-at}\bar{v}_1$. $\bar{v}_1$ is the direction vector of the line $N+\alpha_1 C = \alpha_1\alpha_2$ so for $a<1$ solutions approach this line tangentially near $(\bar{N}_1, \bar{C}_1)$

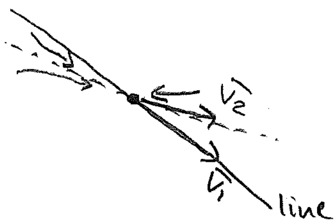
In the Maple plot $a=\frac{1}{2}$ so it happens there.

If $a>1$ (e.g. if $\alpha_1 = 2$ and $\alpha_2 > 3$), $e^{-at} \to 0$ faster than e^{-t} and solutions approach $(\bar{N}_1, \bar{C}_1)$ tangentially with direction $\bar{v}_2$

$a<1$



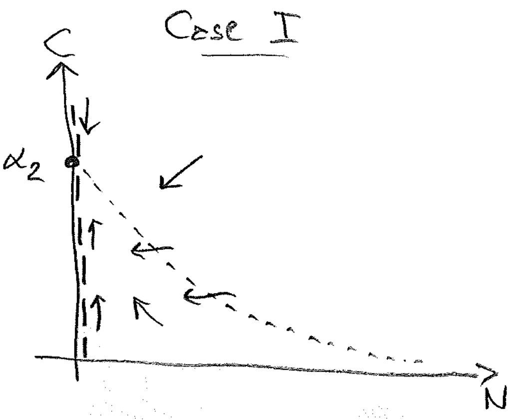
$a>1$

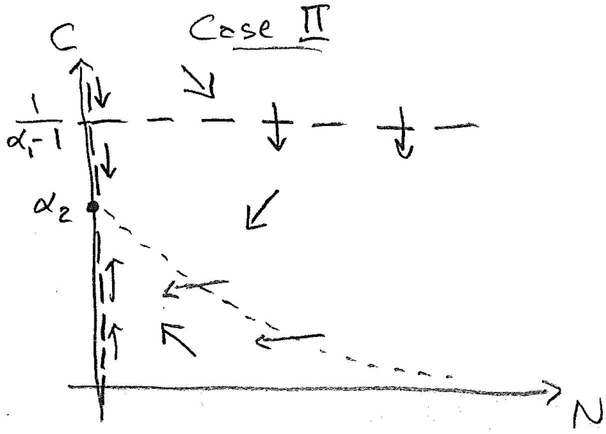


(we cannot cross the line)

Phase plane for chemostat if $\underbrace{\alpha_1<1}_{\text{case I}}$ or $\underbrace{\alpha_2<\frac{1}{\alpha_1-1}}_{\text{case II}}$.

Of the N nullclines, $N=0$ remains but in case I, $C=\frac{1}{\alpha_1-1}$ disappears (we want $C\geq 0$)
The C nullcline $N=\frac{\alpha_2}{C}+\alpha_2-1-C$ stays.





In both cases $(\bar N_2,\bar C_2)=(0,\alpha_2)$ is the only steady state, and it is stable (see sem. 5)

$\lambda_2=-1$ and $|\lambda_1| = \dfrac{1+\overbrace{(1-\alpha_1)}^{\in(0,1)\text{ case I}}\alpha_2}{1+\alpha_2} = \dfrac{1-\overbrace{\alpha_2(\alpha_1-1)}^{<1\text{ case II}}}{1+\alpha_2} < 1 = |\lambda_2| \Rightarrow$

$e^{\lambda_1 t}\to 0$ slower than e^{-t}, $t\to\infty$, in both cases $\Rightarrow$
solutions approach $(\bar N_2,\bar C_2)$ along eigenvector $\bar v_1=\binom{\alpha_1}{-1}$ of λ_1,
this is again tangentially to the line $N+\alpha_1 C=\alpha_1\alpha_2$.

See Maple plots of phase plane and curve $N(t)$ and $C(t)$ for

case I: $\alpha_1=0.8<1$, $\alpha_2=2$ (page 8.6)

case II: $\alpha_1=1.4$, $\alpha_2=2<\frac{1}{\alpha_1-1}=2.5$ (page 8.7)

Chemostat $\alpha_1 = 2$, $\alpha_2 = 2$

> with(DEtools) :

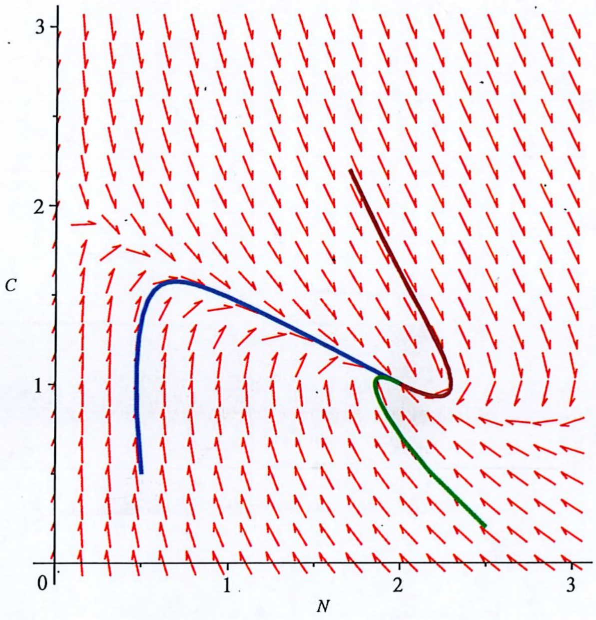
>

> $sys := \left\{ diff(N(t), t) = \frac{2 \cdot N(t) \cdot C(t)}{(1 + C(t))} - N(t),\ diff(C(t), t) = -\frac{N(t) \cdot C(t)}{(1 + C(t))} - C(t) + 2 \right\}$

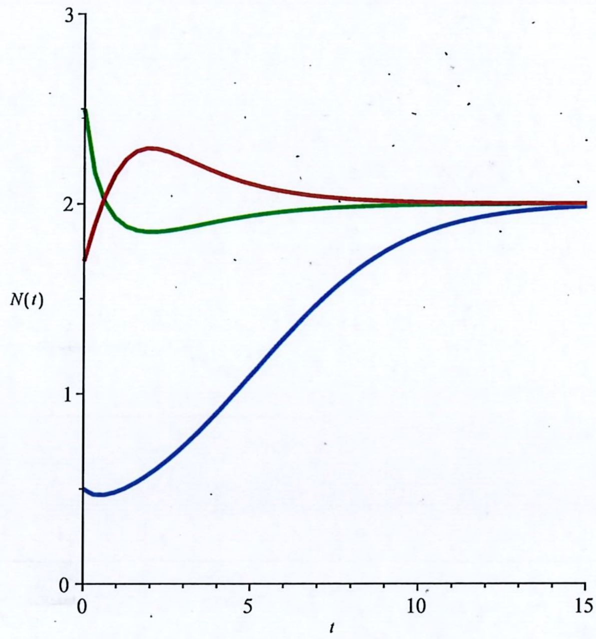
$$sys := \left\{ \frac{d}{dt} C(t) = -\frac{N(t)\, C(t)}{1 + C(t)} - C(t) + 2,\ \frac{d}{dt} N(t) = \frac{2\, N(t)\, C(t)}{1 + C(t)} - N(t) \right\} \quad (1)$$

>

DEplot(sys, [N(t), C(t)], t = 0..15, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], N = 0..3, C = 0..3, linecolor = [blue, green, brown])

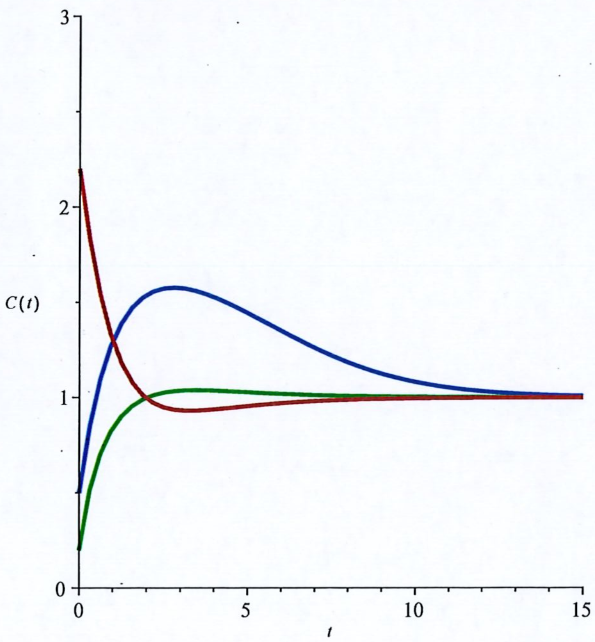


DEplot(sys, [N(t), C(t)], t = 0..15, N = 0..3, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], scene = [t, N(t)], linecolor = [blue, green, brown])



>

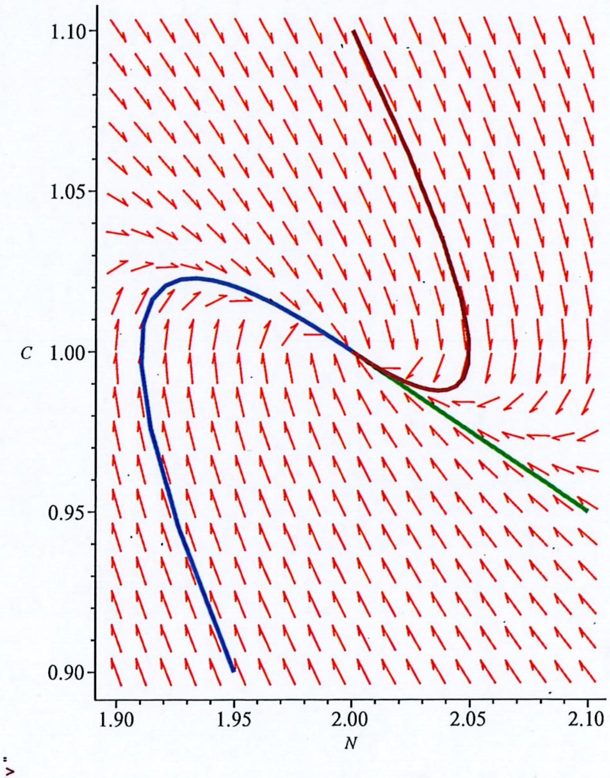
DEplot(sys, [N(t), C(t)], t = 0..15, C = 0..3, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], scene = [t, C(t)], linecolor = [blue, green, brown])



>

(zoomed in)

DEplot(sys, [N(t), C(t)], t = 0..15, [[N(0) = 1.95, C(0) = 0.9], [N(0) = 2.1, C(0) = 0.95], [N(0) = 2.0, C(0) = 1.1]], N = 1.9..2.1, C = 0.9..1.1, linecolor = [blue, green, brown])



>

Chemostat

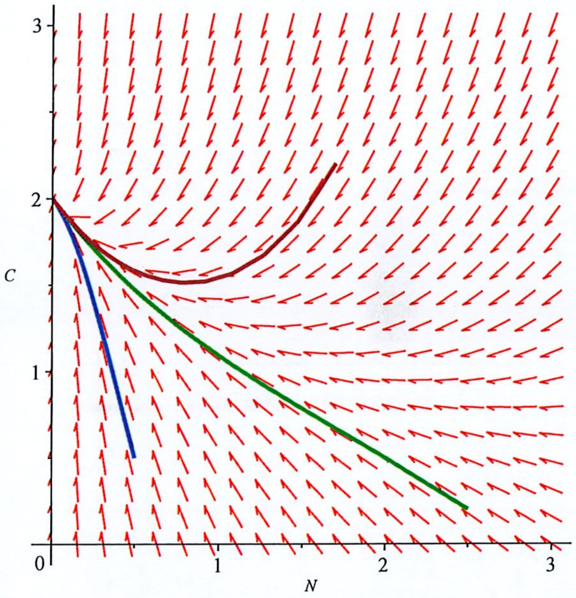
$\alpha_1 = 0.8 < 1$, $\alpha_2 = 2$

> with(DEtools) :

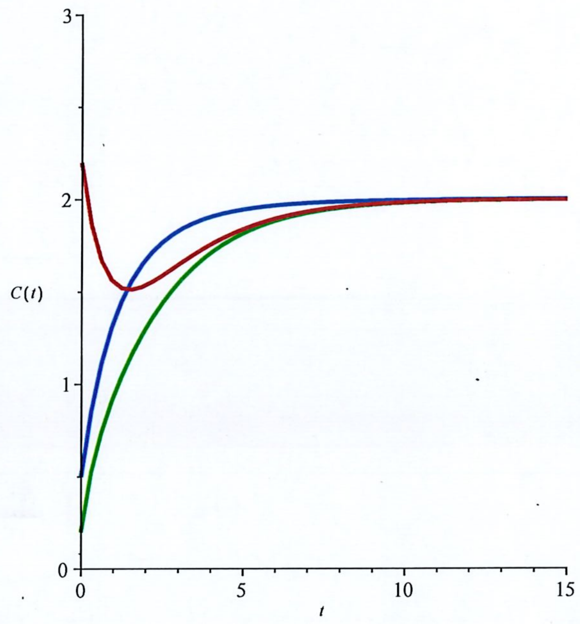
> $sys := \left\{ diff(N(t), t) = \frac{0.8 \cdot N(t) \cdot C(t)}{(1 + C(t))} - N(t), diff(C(t), t) = -\frac{N(t) \cdot C(t)}{(1 + C(t))} - C(t) + 2 \right\}$

$$sys := \left\{ \frac{d}{dt} C(t) = -\frac{N(t)\, C(t)}{1 + C(t)} - C(t) + 2, \frac{d}{dt} N(t) = \frac{0.8\, N(t)\, C(t)}{1 + C(t)} - N(t) \right\} \quad (1)$$

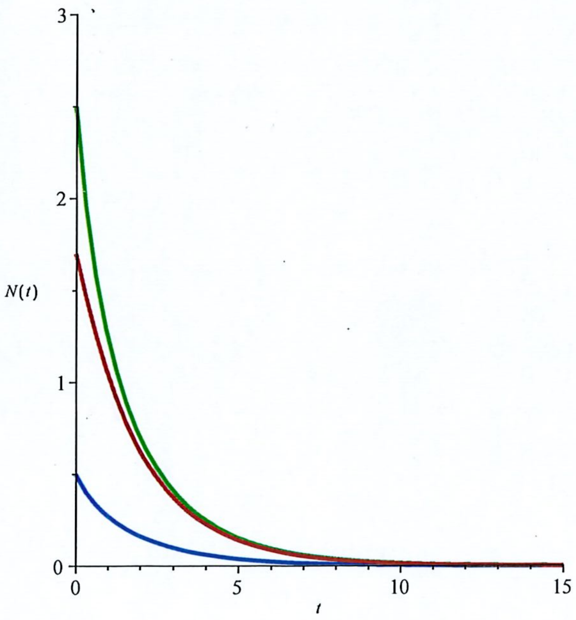
> DEplot(sys, [N(t), C(t)], t = 0..15, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], N = 0..3, C = 0..3, linecolor = [blue, green, brown])



DEplot(sys, [N(t), C(t)], t = 0..15, C = 0..3, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], scene = [t, C(t)], linecolor = [blue, green, brown])



DEplot(sys, [N(t), C(t)], t = 0..15, N = 0..3, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], scene = [t, N(t)], linecolor = [blue, green, brown])



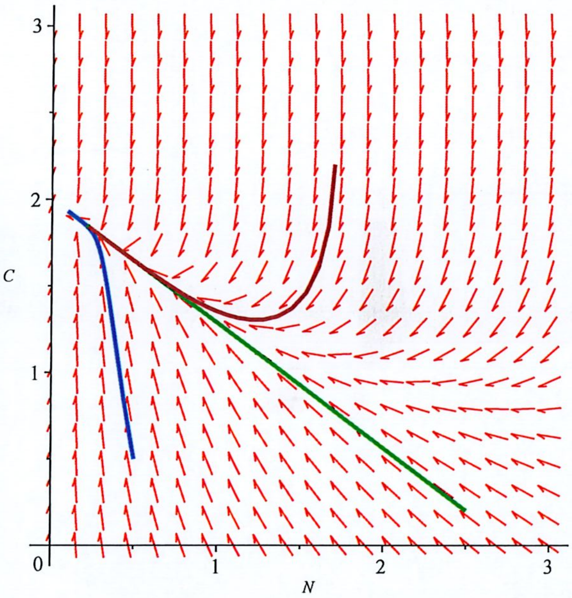
Chemostat

$\alpha_1 = 1.4 \quad , \alpha_2 = 2 < \frac{1}{\alpha_1 - 1} = 2.5$

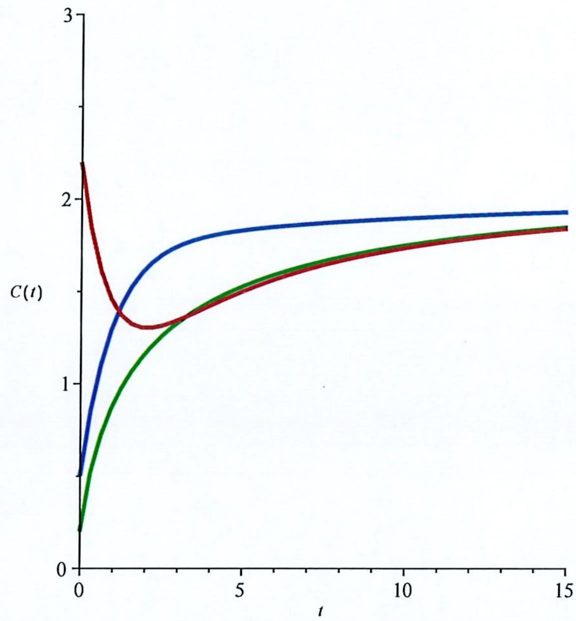
```
> with(DEtools):
>
> sys := {diff(N(t), t) = 1.4·N(t)·C(t)/(1 + C(t)) − N(t), diff(C(t), t) = −N(t)·C(t)/(1 + C(t)) − C(t) + 2}
```

$$sys := \left\{ \frac{d}{dt} C(t) = -\frac{N(t)\,C(t)}{1 + C(t)} - C(t) + 2,\ \frac{d}{dt} N(t) = \frac{1.4\,N(t)\,C(t)}{1 + C(t)} - N(t) \right\} \quad (1)$$

```
> DEplot(sys, [N(t), C(t)], t = 0..15, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], N = 0..3, C = 0..3, linecolor = [blue, green, brown])
```




```
DEplot(sys, [N(t), C(t)], t = 0..15, C = 0..3, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], scene = [t, C(t)], linecolor = [blue, green, brown])
```




```
DEplot(sys, [N(t), C(t)], t = 0..15, N = 0..3, [[N(0) = 0.5, C(0) = 0.5], [N(0) = 2.5, C(0) = 0.2], [N(0) = 1.7, C(0) = 2.2]], scene = [t, N(t)], linecolor = [blue, green, brown])
```