Linköpings universitet
Matematiska institutionen

**Kurskod: TATM38**
**Provkod: TEN1**

**Written examination, TATM38 Mathematical Models in Biology**

**2021-08-23 , 8.00 - 13.00**

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

**1.** The interaction between two species $x(t)$ and $y(t)$ is modeled by the linear system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

Determine the general solution to this system and draw a picture of the entire phase plane (including negative $x$ and $y$). Is the steady state (= equilibrium point) $(0, 0)$ stable?
What is the limit of $\dfrac{x(t)}{y(t)}$ as $t \to \infty$ if $x(0) > 0$ and $y(0) > 0$?
What is the solution if $x(0) = y(0) = 1$?

**2.** Consider the time discrete model

$$x_{n+1} = \frac{2x_n}{1 + x_n^2} \quad , \; n = 0, 1, 2, \ldots$$

Find the steady states (= equilibrium points) and determine their stability. What happens to $x_n$ as $n \to \infty$ in the cases $x_0 > 0$ and $x_0 < 0$? Sketch a cobweb diagram for some $x_0 > 0$.

**3.** A model for interacting whale and krill populations in the sea is given by

$$\begin{cases} \dfrac{dx}{dt} = 4x\left(1 - \dfrac{x}{4}\right) - xy \\ \dfrac{dy}{dt} = \dfrac{y}{2}\left(1 - \dfrac{y}{2 + x}\right) \end{cases}$$

Here $x(t)$ is the whale population and $y(t)$ the krill population (not measured in the same units!). Find all steady states and determine their stability. Draw, for $x \geq 0$ and $y \geq 0$, a phase plane picture (with nullclines and directions of the vector field). What happens to the populations as $t \to \infty$ if $x(0) > 0$ and $y(0) > 0$?

**PLEASE TURN**

**4.** A time discrete predator-prey model is given by the system

$$\begin{cases} x_{n+1} = x_n + 2x_n(1-x_n) - 3x_n y_n \\ y_{n+1} = \frac{2}{3} y_n + x_n y_n \end{cases}$$

Here $x_n$ and $y_n$ are the numbers of prey and predators, respectively, at time $n$. Find all steady states and determine their stability.

**5.** For $0 < x < 1$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t,x)$

$$\begin{cases} u_t = u_{xx} - 2u_x \\ u(t,0) = 0 \\ u(t,1) = 0 \\ u(0,x) = e^x \sin 3\pi x \end{cases}$$

Hint: put $u(t,x) = v(t,x)e^{\alpha t + \beta x}$.

**6.** Let $u(x,t)$ be activator concentration and $v(x,t)$ inhibitor concentration in a Gierer-Meinhardt activator-inhibitor model for animal coat pattern formation, which for one space dimension is given by

$$\begin{cases} u_t = \frac{1}{9} - \frac{10}{9}u + \frac{u^2}{v} + D_1\, u_{xx} \\ v_t = u^2 - v + D_2\, v_{xx} \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability.
Suppose now that $D_1 = 1$, $D_2 = 9$, that $0 < x < L = 6\pi$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t} \cos qx$ with $q = n\pi/L = n/6$, $n = 0, 1, 2, \ldots$ For what values of $n$ can we have diffusive instabilities ($\sigma > 0$) ? Sketch the resulting patterns.

# TATM 38, 23/8 2021, solution sketches

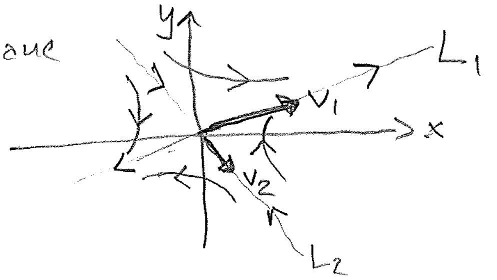
① $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ Eigenvalues $\begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$

$\left.\begin{matrix} \lambda_1 > 0 \\ \lambda_2 < 0 \end{matrix}\right\} \Rightarrow$ saddle point (unstable) at $(0,0)$

Eigenvectors $\lambda_1 = 2 \Rightarrow v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\lambda_2 = -2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$ general solution is

$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Phase plane




Solutions with $x(0)>0$, $y(0)>0$ will approach the line $L_1$ parallel to $\bar{v}_1$ for large $t$ ($L_1$ is $y = x/3$)

$e^{-2t} \to 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \approx c_1 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow$

$\frac{x(t)}{y(t)} \to 3$, $t \to \infty$ (unless $c_1 = 0$ which is for initial values on $L_2$)

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{1}{2}e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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② $x_{n+1} = \dfrac{2x_n}{1+x_n^2} = f(x_n)$

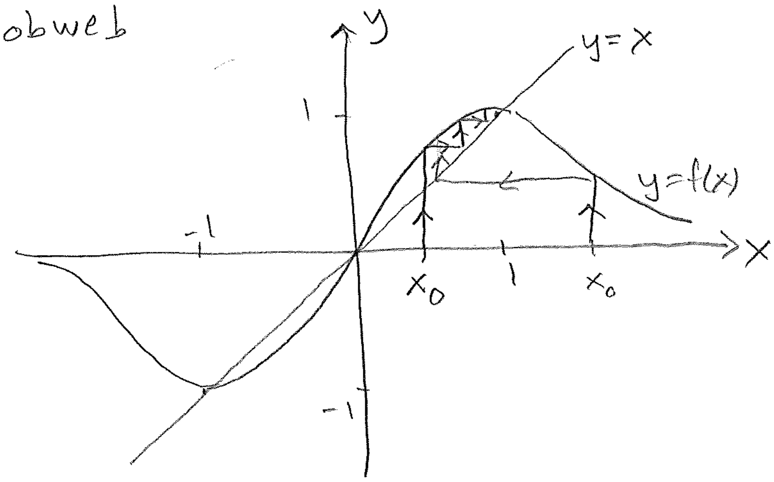
Steady states $\bar{x} = f(\bar{x}) = \dfrac{2\bar{x}}{1+\bar{x}^2} \Rightarrow \bar{x}_1 = 0$ or $1+\bar{x}^2 = 2 \Rightarrow \bar{x}_{2,3} = \pm 1$ 3 steady states

$f'(x) = \dfrac{2(1+x^2) - 2x \cdot 2x}{(1+x^2)^2} = \dfrac{2-2x^2}{(1+x^2)^2} \Rightarrow |f'(\bar{x}_1)| = |f'(0)| = 2 > 1$ unstable

$|f'(\bar{x}_2)| = |f'(1)| = 0 < 1$ stable

$|f'(\bar{x}_3)| = |f'(-1)| = 0 < 1$ stable

Cobweb




$x_n \to 1$ as $n \to \infty$ if $x_0 > 0$

(and $x_n \to -1$ if $x_0 < 0$)

($f' = 0$ at $x = \pm 1$, $f \to 0, x \to \pm\infty$)

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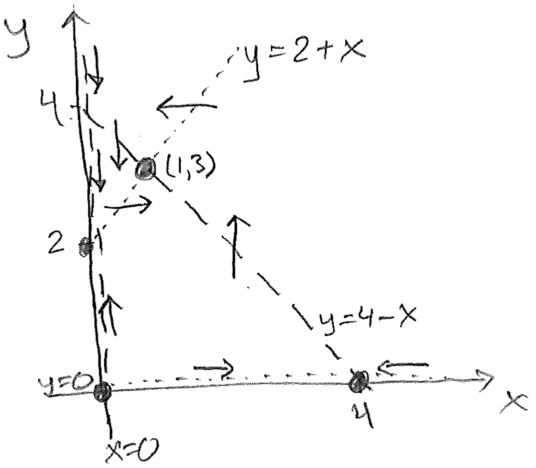
(3) $\begin{cases} x' = 4x\left(1-\frac{x}{4}\right) - xy = x(4-x-y) & x \text{ nullclines}: x=0,\ y=4-x \quad 2 \text{ lines} \\ y' = \frac{y}{2}\left(1-\frac{y}{2+x}\right) & y \text{ nullclines}: y=0,\ y=2+x \quad 2 \text{ lines} \end{cases}$

Phase plane (for $x \geq 0, y \geq 0$)




4 steady states $(\bar{x}_1,\bar{y}_1)=(0,0)$, $(\bar{x}_2,\bar{y}_2)=(4,0)$
$(\bar{x}_3,\bar{y}_3)=(1,3)$, $(\bar{x}_4,\bar{y}_4)=(0,2)$

$$J(x,y) = \begin{pmatrix} 4-2x-y & -x \\ \frac{y^2}{2(2+x)^2} & \frac{1}{2}-\frac{y}{2+x} \end{pmatrix} \qquad J(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \begin{cases} \lambda_1 = 4 > 0 \\ \lambda_2 = 1/2 > 0 \end{cases} \text{unstable}$$

$$J(4,0) = \begin{pmatrix} -4 & -4 \\ 0 & 1/2 \end{pmatrix} \quad \begin{matrix} \lambda_1 = -4 < 0 \\ \lambda_2 = 1/2 > 0 \end{matrix} \ \text{unstable (saddle)} \qquad J(0,2) = \begin{pmatrix} 2 & 0 \\ 1/2 & -1/2 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 2 > 0 \\ \lambda_2 = -1/2 < 0 \end{matrix} \ \text{unstable (saddle)}$$

$$J_3 = J(1,3) = \begin{pmatrix} -1 & -1 \\ 1/2 & -1/2 \end{pmatrix} \quad \begin{matrix} \operatorname{Tr} J_3 = -\frac{3}{2} < 0 \\ \det J_3 = 1 > 0 \end{matrix} \Rightarrow \underline{\text{stable}} \quad \left[\begin{matrix} \text{or } \lambda^2 + \frac{3}{2}\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = -\frac{3}{4} \pm i\frac{\sqrt{7}}{4}. \\ \operatorname{Re}\lambda_{1,2} < 0 \Rightarrow \text{stable (spiral)} \end{matrix}\right]$$

The populations approach the stable steady state $(\bar{x}_3,\bar{y}_3) = (1,3)$ as $t \to \infty$

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(4) $\begin{cases} x_{n+1} = x_n + 2x_n(1-x_n) - 3x_n y_n \\ y_{n+1} = \frac{2}{3} y_n + x_n y_n \end{cases}$ Steady states $\begin{rcases} \bar{x} = x_{n+1} = x_n \\ \bar{y} = y_{n+1} = y_n \end{rcases} \Rightarrow$

$$\begin{cases} \bar{x} = \bar{x} + 2\bar{x}(1-\bar{x}) - 3\bar{x}\bar{y} \\ \bar{y} = \frac{2}{3}\bar{y} + \bar{x}\bar{y} \end{cases} \Rightarrow \begin{cases} \bar{x}(2-2\bar{x}-3\bar{y}) = 0 & (1) \\ \bar{y}\left(\bar{x}-\frac{1}{3}\right) = 0 & (2) \end{cases} \Rightarrow \bar{y} = 0 \ \text{or} \ \bar{x} = \frac{1}{3}$$

$\bar{y} = 0 \overset{(1)}{\Rightarrow} \bar{x} = 0$ or $\bar{x} = 1$; $\quad \bar{x} = \frac{1}{3} \overset{(1)}{\Rightarrow} \bar{y} = \frac{4}{9}$

$\Rightarrow$ 3 steady states $(\bar{x}_1,\bar{y}_1) = (0,0)$, $(\bar{x}_2,\bar{y}_2) = (1,0)$, $(\bar{x}_3,\bar{y}_3) = \left(\frac{1}{3},\frac{4}{9}\right)$

$$J(x,y) = \begin{pmatrix} 3-4x-3y & -3x \\ y & x+\frac{2}{3} \end{pmatrix}, \quad J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2/3 \end{pmatrix} \quad |\lambda_1| = |3| > 1 \Rightarrow \underline{\text{unstable}} \quad \left(|\lambda_2| = \left|\tfrac{2}{3}\right| < 1\right)$$

$$J(1,0) = \begin{pmatrix} -1 & -3 \\ 0 & 5/3 \end{pmatrix}, \quad \begin{matrix} |\lambda_1| = |-1| = 1 \\ |\lambda_2| = \left|\frac{5}{3}\right| = \frac{5}{3} > 1 \end{matrix} \Rightarrow \underline{\text{unstable}}$$

$$J_3 = J\left(\tfrac{1}{3},\tfrac{4}{9}\right) = \begin{pmatrix} 1/3 & -1 \\ 4/9 & 1 \end{pmatrix} \quad \lambda^2 - \frac{4}{3}\lambda + \frac{7}{9} = 0, \quad \lambda_{1,2} = \frac{2}{3} \pm \sqrt{\frac{4}{9}-\frac{7}{9}} = \frac{2 \pm i\sqrt{3}}{3} \Rightarrow |\lambda_{1,2}| = \sqrt{\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{\sqrt{3}}{3}\right)^2} =$$

$$= \sqrt{\frac{7}{9}} < 1 \Rightarrow \underline{\text{stable}} \quad \left(\text{with Jury test } \underbrace{|\operatorname{Tr} J_3|}_{4/3} < \underbrace{1 + \underbrace{\det J_3}_{7/9}}_{16/9} < 2 \ \text{ satisfied!} \Rightarrow \text{stable}\right)$$

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(5) $u(t,x)=v(t,x)e^{\alpha t+\beta x} \Rightarrow u_t=(v_t+\alpha v)e^{\alpha t+\beta x},\ u_x=(v_x+\beta v)e^{\alpha t+\beta x},$

$u_{xx}=(v_{xx}+2\beta v_x+\beta^2 v)e^{\alpha t+\beta x}$. Substitute into $u_t=u_{xx}-2u_x \Rightarrow$

$(v_t+\underline{\underline{\alpha v}})\cancel{e^{\alpha t+\beta x}}=(v_{xx}+\underline{2\beta v_x}+\underline{\underline{\beta^2 v}}-\underline{2v_x}-\underline{\underline{2\beta v}})\cancel{e^{\alpha t+\beta x}}$ With $\beta=1$ and

$\alpha=\beta^2-2\beta=-1$ we get $v_t=v_{xx}$ and IBVP for $v(t,x)$:

$$\begin{cases} v_t=v_{xx} \\ v(t,0)=u(t,0)e^{-\alpha t-\beta 0}=0 \\ v(t,1)=u(t,1)e^{-\alpha t-\beta\cdot 1}=0 \\ v(0,x)=u(0,x)e^{-\beta x}=e^x\sin 3\pi x\cdot e^{-x}=\sin 3\pi x \end{cases}$$

$v(t,x)=T(t)X(x) \Rightarrow$ (separation of variables)

$\frac{T'}{T}=\frac{X''}{X}=\text{constant}=\lambda$

and $X(0)=X(1)=0$

Solutions to $X$-eq. $X_n(x)=\sin n\pi x,\ n=1,2,3,\ldots$

$\lambda=-n^2\pi^2$ T-eq $T'=-n^2\pi^2 T \Rightarrow T_n(t)=e^{-n^2\pi^2 t} \Rightarrow v(t,x)=\sum_{n=1}^{\infty}a_n e^{-n^2\pi^2 t}\sin n\pi x$

$v(0,x)=\sum_{n=1}^{\infty}a_n e^0\sin n\pi x=\sin 3\pi x \Rightarrow a_3=1,\ a_n=0$ for $n\neq 3 \Rightarrow$

$v(t,x)=e^{-9\pi^2 t}\sin 3\pi x \Rightarrow \underline{u(t,x)}=v(t,x)e^{-t+x}=\underline{e^{-(1+9\pi^2)t}e^x\sin 3\pi x}$

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(6) $$\begin{cases} u_t=\frac{1}{9}-\frac{10}{9}u+\frac{u^2}{v}+D_1u_{xx} \\ v_t=u^2-v+D_2v_{xx} \end{cases}$$

Spatially uniform steady state $(\bar u,\bar v)$:

$$\begin{cases} \frac{1}{9}-\frac{10}{9}\bar u+\frac{\bar u^2}{\bar v}=0 \\ \bar u^2-\bar v=0 \end{cases} \Rightarrow (\bar u,\bar v)=(1,1)$$

Stability for $D_1=D_2=0$:

Jacobian $J(u,v)=\begin{pmatrix} -\frac{10}{9}+\frac{2u}{v} & -\frac{u^2}{v^2} \\ 2u & -1 \end{pmatrix} \Rightarrow J(1,1)=\begin{pmatrix} 8/9 & -1 \\ 2 & -1 \end{pmatrix}=J$ $\begin{cases} \operatorname{Tr}J=-1/9<0 \\ \det J=\frac{10}{9}>0 \end{cases} \Rightarrow$ stable

Turing condition $J_{11}D_2+J_{22}D_1>2\sqrt{D_1D_2\cdot\det J} \Leftrightarrow \underline{\frac{8}{9}D_2-D_1>2\sqrt{10D_1D_2/9}}$

With $D_1=1,\ D_2=9$ $\quad 8-1>2\sqrt{10}$ (satisfied)

Unstable perturbations $(q=\frac{n}{6})$: $0>\det(J-q^2D)=\begin{vmatrix} 8/9-1\cdot q^2 & -1 \\ 2 & -1-9q^2 \end{vmatrix}=$

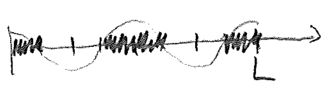
$=9q^4-7q^2+\frac{10}{9}=(3q^2-\frac{7}{6})^2-\frac{9}{36} \Leftrightarrow |3q^2-\frac{7}{6}|<\sqrt{\frac{9}{36}}=\frac{1}{2} \Leftrightarrow -\frac{1}{2}<3q^2-\frac{7}{6}<\frac{1}{2}$

$\Leftrightarrow \frac{4}{6}<3q^2<\frac{10}{6} \Leftrightarrow \frac{2}{9}<q^2<\frac{5}{9} \Leftrightarrow \frac{2}{9}<\frac{n^2}{36}<\frac{5}{9} \Leftrightarrow 8<n^2<20$

$\Rightarrow n=3$ or $n=4$

Patterns


$n=3,\ \cos\frac{3\pi x}{L}$




$n=4,\ \cos\frac{4\pi x}{L}$

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