Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology
2021-10-30 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. (a) The interaction between two species $x(t)$ and $y(t)$ is modeled by the linear system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1/3 & \alpha \\ \alpha & -1/3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} ,$$

where $\alpha > 0$ is a constant. For what values of α is the steady state (= equilibrium point) $(0, 0)$ stable? Determine the general solution to this system.

(b) Consider the linear system of difference equations with the same matrix (and again $\alpha > 0$ constant)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} -1/3 & \alpha \\ \alpha & -1/3 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} .$$

For what values of α is $(0, 0)$ stable in this case? Give the general solution to the system.

2. A population $N(t)$ is described by a logistic equation with a fishing term $4N(1-N)^2$:

$$\frac{dN}{dt} = 2N(1-N) - 4N(1-N)^2$$

Find all steady states (= equilibrium points) and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$ for different initial values $N(0)$?

3. Two populations $N(t)$ and $P(t)$ compete for the same resources, and their time evolution is given by

$$\begin{cases} \dfrac{dN}{dt} = N(1-N-3P) \\ \dfrac{dP}{dt} = 2P(1-P-2N) \end{cases}$$

Find all steady states and determine their stability. Draw a phase plane picture with nullclines and directions of the vector field. What can happen to the populations as $t \to \infty$? Give an interpretation of what the result means for the possibility of coexistence.

PLEASE TURN

4. Let S_n, I_n and R_n be the number of susceptibles, infective, and removed, respectively, at time $n \geq 0$ in a time discrete SIRS epidemic model. The total population N is constant, $N = 900$, and with $R_n = N - S_n - I_n$ one has a two-dimensional system for S_n and I_n:

$$\begin{cases} S_{n+1} = S_n - \dfrac{S_n I_n}{1000} + \dfrac{1}{10}(900 - S_n - I_n) \\ I_{n+1} = I_n + \dfrac{S_n I_n}{1000} - \dfrac{2I_n}{5} \end{cases}$$

Find all steady states (= equilibrium points) and determine their stability.

5. For $0 < x < \pi/2$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} 2u_t = u_{xx} + 2u \\ u_x(t, 0) = 0 \\ u_x(t, \pi/2) = 0 \\ u(0, x) = 3\cos 4x - \cos 2x \end{cases}$$

Hint: put $u(t, x) = v(t, x)e^{\alpha t}$.

6. A substrate depletion (or positive feedback) model for two chemical concentrations $u(x, t)$ and $v(x, t)$ of a type originally studied by Alan Turing is given by

$$\begin{cases} u_t = uv - u - 2 + D_1\, u_{xx} \\ v_t = 4 - uv + D_2\, v_{xx} \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability. Suppose now that $D_1 = 1/30$, $D_2 = 1/2$, that $0 < x < L = \sqrt{3}\pi$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t} \cos qx$ with $q = n\pi/L = n/\sqrt{3}$, $n = 0, 1, 2, \ldots$ For what values of n can we have diffusive instabilities ($\sigma > 0$)? Sketch the resulting patterns.

TATM38, Math. Bio, 30/10 2021, solution sketches

① a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1/3 & \alpha \\ \alpha & -1/3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$ Eigenvalues $\begin{vmatrix} -1/3-\lambda & \alpha \\ \alpha & -1/3 \end{vmatrix} = (\lambda + \frac{1}{3})^2 - \alpha^2 = 0 \Rightarrow \lambda_{1,2} = -\frac{1}{3} \pm \alpha$ (real)

(0,0) stable if $\lambda_{1,2} < 0$ $\begin{cases} -1/3 + \alpha < 0 \Rightarrow \alpha < \frac{1}{3} \\ -1/3 - \alpha < 0 \text{ for all } \alpha > 0 \end{cases} \Rightarrow$ stable if $\alpha < \frac{1}{3}$ ($\alpha > 0$ was assumed)

[can also check $\begin{cases} \mathrm{Tr}\,A = -2/3 < 0 \\ \det A = 1/9 - \alpha^2 > 0 \Rightarrow \alpha^2 < 1/9 \Rightarrow \alpha < 1/3 \end{cases}$]

Eigenvectors

$\lambda_1: \left(\begin{array}{cc|c} -\alpha & \alpha & 0 \\ \alpha & -\alpha & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2: \left(\begin{array}{cc|c} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution is $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2 = c_1 e^{(-\frac{1}{3}+\alpha)t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{(-\frac{1}{3}-\alpha)t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b) $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, (0,0) stable if $|\lambda_{1,2}| < 1$ $\begin{cases} |-1/3+\alpha| < 1 \Rightarrow -1 < -\frac{1}{3} + \alpha < 1 \Rightarrow -\frac{2}{3} < \alpha < \frac{4}{3} \\ |-1/3-\alpha| = |\frac{1}{3}+\alpha| < 1 \Rightarrow -1 < \frac{1}{3} + \alpha < 1 \Rightarrow -\frac{4}{3} < \alpha < \frac{2}{3} \end{cases}$

$\Rightarrow$ stable if $\alpha < \frac{2}{3}$ ($\alpha > 0$ assumed)

[or check with Jury test $\underbrace{|\mathrm{Tr}\,A|}_{2/3} < 1 + \underbrace{\det A}_{1/9-\alpha^2} < 2$ ← satisfied

$\Rightarrow \frac{2}{3} < \frac{1}{9} - \alpha^2 \Rightarrow \alpha^2 < \frac{4}{9} \Rightarrow \alpha < \frac{2}{3}$]

General solution is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 \lambda_1^n \bar{v}_1 + c_2 \lambda_2^n \bar{v}_2 = c_1 (-\frac{1}{3}+\alpha)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (-\frac{1}{3}-\alpha)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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② $\frac{dN}{dt} = f(N) = 2N(1-N) - 4N(1-N)^2 = 2N(1-N)[1-2(1-N)] = 2N(1-N)(2N-1)$

Steady states: $f(\bar{N}) = 0 \Rightarrow \bar{N}_1 = 0, \bar{N}_2 = 1/2, \bar{N}_3 = 1$, stable if $f'(N_j) < 0$

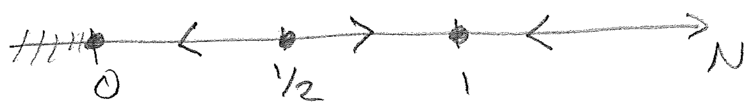
$f'(N) = 2(1-N)(2N-1) - 2N(2N-1) + 4N(1-N) \Rightarrow$

$f'(0) = -2 < 0 \Rightarrow \bar{N}_1 = 0$ stable

$f'(1/2) = 1 > 0 \Rightarrow \bar{N}_2 = 1/2$ unstable

$f'(1) = -2 < 0 \Rightarrow \bar{N}_3 = 1$ stable

Phase line for $N \geq 0$



$N(t) \to \begin{cases} 0 & \text{if } N(0) < 1/2 \\ 1/2 & \text{if } N(0) = 1/2 \\ 1 & \text{if } N(0) > 1/2 \end{cases}$

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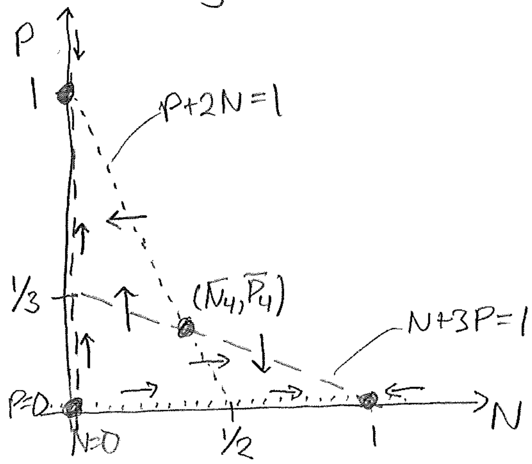
③ $\begin{cases} N' = N(1-N-3P) \\ P' = 2P(1-P-2N) \circledast \end{cases}$ N-nullclines $N=0$ and $N+3P=1$ (dashed)
P-nullclines $P=0$ and $P+2N=1$ (dotted)

steady states $N=0$ in $\circledast$ $\Rightarrow P(1-P)=0 \Rightarrow P=0$ or $P=1$

$N=1-3P$ in $\circledast \Rightarrow P(5P-1)=0 \Rightarrow P=0, N=1$ or $P=1/5, N=2/5$

Four steady states $(\bar{N}_1,\bar{P}_1)=(0,0)$, $(\bar{N}_2,\bar{P}_2)=(1,0)$, $(\bar{N}_3,\bar{P}_3)=(0,1)$, $(\bar{N}_4,\bar{P}_4)=\left(\frac{2}{5},\frac{1}{5}\right)$



$$J(N,P)=\begin{pmatrix} 1-2N-3P & -3N \\ -4P & 2-4P-4N \end{pmatrix} \Rightarrow$$

$J(0,0)=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\left.\begin{matrix}\lambda_1=1>0 \\ \lambda_2=2>0\end{matrix}\right\} \Rightarrow (0,0)$ unstable

$J(1,0)=\begin{pmatrix} -1 & -3 \\ 0 & -2 \end{pmatrix}$ $\left.\begin{matrix}\lambda_1=-1<0 \\ \lambda_2=-2<0\end{matrix}\right\} \Rightarrow (1,0)$ stable

$J(0,1)=\begin{pmatrix} -2 & 0 \\ -4 & -2 \end{pmatrix}$ $\left.\begin{matrix}\lambda_1=-2<0 \\ \lambda_2=-2<0\end{matrix}\right\} \Rightarrow (0,1)$ stable

$J\left(\frac{2}{5},\frac{1}{5}\right)=\begin{pmatrix} -2/5 & -6/5 \\ -4/5 & -2/5 \end{pmatrix} \Rightarrow \lambda_{1,2}=-\frac{2}{5}\pm\frac{\sqrt{24}}{5}$ $\begin{cases}\lambda_1>0 \\ \lambda_2<0\end{cases}$ $\Rightarrow$ saddle, unstable
[or check $\det J=\frac{4}{25}-\frac{24}{25}=-\frac{4}{5}<0 \Rightarrow$ saddle]

As $t\to\infty$, (N,P) can approach either $(1,0)$ or $(0,1)$, which one depends on initial values $(N(0),P(0))$. Only one population survives, coexistence is not possible.

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④ $\begin{cases} S_{n+1}=S_n-\frac{S_nI_n}{1000}+\frac{1}{10}(900-S_n-I_n) \\ I_{n+1}=I_n+\frac{S_nI_n}{1000}-\frac{2}{5}I_n \end{cases}$

Steady states $\bar{S}=S_{n+1}=S_n$, $\bar{I}=I_{n+1}=I_n$

$\Rightarrow \begin{cases} \bar{S}=\bar{S}-\frac{\bar{S}\bar{I}}{1000}+\frac{1}{10}(900-\bar{S}-\bar{I}) \circledast \\ \bar{I}=\bar{I}+\frac{\bar{S}\bar{I}}{1000}-\frac{2}{5}\bar{I} \Rightarrow \bar{I}\left(\frac{\bar{S}}{1000}-\frac{2}{5}\right)=0 \Rightarrow \bar{I}=0 \text{ or } \bar{S}=400 \end{cases}$

$\bar{I}=0$ in $\circledast \Rightarrow \bar{S}=900$, $\bar{S}=400$ in $\circledast \Rightarrow \bar{I}=100$ $\Rightarrow$ Two steady states:
$(\bar{S}_1,\bar{I}_1)=(900,0)$ and $(\bar{S}_2,\bar{I}_2)=(400,100)$ (with $\bar{R}_1=0$ and $\bar{R}_2=400$ resp.)

$J(S,I)=\begin{pmatrix} 1-\frac{I}{1000}-\frac{1}{10} & -\frac{S}{1000}-\frac{1}{10} \\ \frac{I}{1000} & 1+\frac{S}{1000}-\frac{2}{5} \end{pmatrix} \Rightarrow J(900,0)=\begin{pmatrix} 0.9 & -1 \\ 0 & 1.5 \end{pmatrix} \Rightarrow \begin{cases}\lambda_1=0.9 \\ \lambda_2=1.5\end{cases}$

$|\lambda_2|=1.5>1 \Rightarrow (\bar{S}_1,\bar{I}_1)$ unstable

$J(400,100)=\begin{pmatrix} 0.8 & -0.5 \\ 0.1 & 1 \end{pmatrix} \Rightarrow \lambda^2-1.8\lambda+0.85=0 \Rightarrow \lambda_{1,2}=0.9\pm\sqrt{0.81-0.85}=0.9\pm0.2i$ (complex)

$\Rightarrow |\lambda_{1,2}|=\sqrt{0.9^2+0.2^2}=\sqrt{0.85}<1 \Rightarrow (\bar{S}_2,\bar{I}_2)$ stable

[or use Jury test $\underbrace{|\mathrm{Tr}\,J|}_{1.8}<1+\underbrace{\det J}_{0.85}<2$, satisfied $\Rightarrow$ stable]

(5) $u(t,x) = v(t,x)e^{\alpha t} \Rightarrow u_t = (v_t + \alpha v)e^{\alpha t},\ u_x = v_x e^{\alpha t},\ u_{xx} = v_{xx}e^{\alpha t} \Rightarrow$

$2u_t = u_{xx} + 2u \iff \underline{2(v_t + \alpha v)}\cancel{e^{\alpha t}} = (v_{xx} + \underline{2v})\cancel{e^{\alpha t}}$. Take $2\alpha = 2 \Rightarrow \alpha = 1 \Rightarrow$
$2v_t = v_{xx}$

IBVP for $v(t,x)$:

$$\begin{cases} 2v_t = v_{xx} \\ v_x(t,0) = u_x(t,0)e^{-t} = 0 \\ v_x(t,\pi/2) = u_x(t,\pi/2)e^{-t} = 0 \\ v(0,x) = u(0,x)e^{-0} = u(0,x) = 3\cos 4x - \cos 2x \end{cases}$$

Separation of variables
$v(t,x) = T(t)X(x) \Rightarrow$
$$\frac{2T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{constant}$$
$\Rightarrow T(t) = e^{\lambda t/2}$

$v_x(t,0) = v_x(t,\pi/2) = 0 \Rightarrow X'(0) = X'(\pi/2) = 0$

$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X'(0) = X'(\pi/2) = 0 \end{cases} \Rightarrow X_n(x) = \cos\frac{n\pi x}{\pi/2} = \cos 2nx,\ n = 0,1,2,\dots \quad \lambda = -4n^2$
($n=0$ from $\lambda = 0$, $X_0(x)$ constant)

$\Rightarrow v(t,x) = \sum_{n=0}^{\infty} \alpha_n e^{-2n^2 t}\cos 2nx$ (or $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-2n^2 t}\cos 2nx$)

$v(0,x) = \sum_{n=0}^{\infty} \alpha_n \cos 2nx = 3\cos 4x - \cos 2x$, identify $\underset{(n=1)}{\alpha_1 = -1}$, $\underset{(n=2)}{\alpha_2 = 3}$, all other $\alpha_n = 0$

$\Rightarrow v(t,x) = -e^{-2t}\cos 2x + 3e^{-8t}\cos 4x \Rightarrow$

$\underline{u(t,x)} = v(t,x)e^t = \underline{-e^{-t}\cos 2x + 3e^{-7t}\cos 4x}$

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(6) $\begin{cases} u_t = uv - u - 2 + D_1 u_{xx} \\ v_t = 4 - uv + D_2 u_{xx} \end{cases}$

Spatially uniform steady state $(\bar u, \bar v)$:
$\begin{cases} \bar u \bar v - \bar u - 2 = 0 \\ 4 - \bar u \bar v = 0 \end{cases} \Rightarrow (\bar u, \bar v) = (2,2)$

Stability for $D_1 = D_2 = 0$

$J(u,v) = \begin{pmatrix} v-1 & u \\ -v & -u \end{pmatrix} \Rightarrow J(2,2) = \underbrace{\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}}_{=A}$ $\left.\begin{matrix} \mathrm{Tr}A = -1 < 0 \\ \det A = 2 > 0 \end{matrix}\right\} \Rightarrow$ stable (or $\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2}$ have $\mathrm{Re}(\lambda_{1,2}) < 0$)

Condition for Turing instability:

$a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A} \iff \underline{D_2 - 2D_1 > 2\sqrt{2D_1D_2}}$.

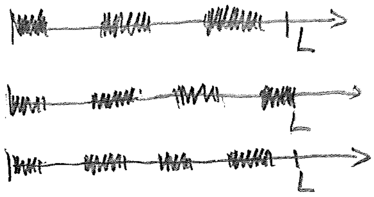
With $D_1 = \frac{1}{30}$, $D_2 = \frac{1}{2}$: $\frac{1}{2} - \frac{2}{30} = \frac{13}{30}$, $2\sqrt{2/60} = \frac{2}{\sqrt{30}} = \frac{2\sqrt{30}}{30} < \frac{13}{30}$ OK ($\sqrt{30} < 6$)

Unstable perturbations ($q = n/\sqrt{3}$): $0 > \det(A - q^2 D) = \begin{vmatrix} 1 - \frac{q^2}{30} & 2 \\ -2 & -2 - \frac{q^2}{2} \end{vmatrix} =$

$= \frac{q^4}{60} - \frac{13q^2}{30} + 2 = \frac{1}{60}(q^4 - 26q^2 + 120) = \frac{1}{60}\left[(q^2-13)^2 - 49\right] \iff (q^2 - 13)^2 < 49 \iff$

$-7 < q^2 - 13 < 7 \iff 6 < q^2 < 20 \iff 6 < \frac{n^2}{3} < 20 \iff \underline{18 < n^2 < 60} \Rightarrow \underline{n = 5, 6, 7}$

Patterns



$n = 5$, $\cos\frac{5\pi x}{L}$

$n = 6$, $\cos\frac{6\pi x}{L}$

$n = 7$, $\cos\frac{7\pi x}{L}$