Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2022-01-08 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. The population $N_n \geq 0$ at time n in a time discrete population model satisfies

$$N_{n+1} = \frac{2}{1+N_n} \quad , \; n = 0, 1, 2, \ldots$$

Find the non-negative steady states (= equilibrium points) of the model and determine their stability. What happens to the population as $n \to \infty$?
Sketch a cobweb diagram (for $N \geq 0$) for some $N_0 > 0$.

2. An ecosystem model for phytoplankton $P(t)$, zooplankton $Z(t)$, and nitrogen leads to the two-dimensional dynamical system for $P(t)$ and $Z(t)$:

$$\begin{cases} \dfrac{dP}{dt} = P(3 - P - 2Z) \\ \dfrac{dZ}{dt} = Z(P - 1) \end{cases}$$

Find all steady states of the system and determine their stability. Draw, for $P \geq 0$ and $Z \geq 0$, a phase plane picture with nullclines and directions of the vector field. What happens to the populations as $t \to \infty$ if $P(0) > 0$ and $Z(0) > 0$?

3. Let u_n, v_n, and w_n denote the frequencies of genotypes AA, AB, and BB, respectively, in a population at generation $n = 0, 1, 2, \ldots$ We have that $u_n + v_n + w_n = 1$ for all n. A model with positive assortative mating gives a linear system for u_n and v_n:

$$\begin{cases} u_{n+1} = u_n + \frac{1}{4} v_n \\ v_{n+1} = \frac{1}{2} v_n \end{cases}$$

Determine the general solution to this system and show that $v_n \to 0$ as $n \to \infty$.
Find u_n, v_n and w_n if we have the initial conditions $u_0 = 0.5$, $v_0 = 0.2$ and $w_0 = 0.3$. What are the asymptotic values of u_n and w_n as $n \to \infty$ in this case?

PLEASE TURN

4. Two competing bacteria populations $x(t)$ and $y(t)$ are grown in a chemostat and modeled by

$$\begin{cases} \dfrac{dx}{dt} = x\Big(5 - \dfrac{6}{2-x-y}\Big) \\ \dfrac{dy}{dt} = y\Big(3 - \dfrac{4}{2-x-y}\Big) \end{cases}$$

where we assume that $x(t) + y(t) \leq 1$. Find all steady states of the system and determine their stability. Draw, for $x \geq 0$, $y \geq 0$ and $x + y \leq 1$, a phase plane picture with nullclines and directions of the vector field. Interpret what the results mean for the two bacteria populations.

5. For $0 < x < 1$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} 3u_t = u_{xx} \\ u(t,0) = 1 \\ u(t,1) = 0 \\ u(0,x) = 1 - x + \sin(\pi x) + 2\sin(3\pi x) \end{cases}$$

Hint: put $u(t,x) = v(t,x) + Ax + B$ to get an IBVP with homogeneous boundary conditions for $v(t,x)$.

6. Consider a reactor-diffusion system in two space dimensions for the functions $u(t,x,y)$ and $v(t,x,y)$

$$\begin{cases} u_t = R_1(u,v) + D_1\,(u_{xx} + u_{yy}) \\ v_t = R_2(u,v) + D_2\,(v_{xx} + v_{yy}) \end{cases}$$

In a spatially uniform steady state $(\bar{u}, \bar{v})$ (this means $R_1(\bar{u},\bar{v}) = R_2(\bar{u},\bar{v}) = 0$), for which of the following three Jacobi matrices $J(\bar{u},\bar{v})$ can we have Turing diffusive instabilities?

$$J_1 = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \qquad J_3 = \begin{pmatrix} -1 & 3 \\ -1 & -2 \end{pmatrix}$$

Justify your answer clearly.

For the matrix that allows diffusive instability, assume now that $0 < x < 3\pi$, $0 < y < 6\pi$, $D_1 = 1$ and $D_2 = 8$. Find the values of (m,n) for which unstable (pattern forming) modes $e^{\sigma t} \cos \frac{mx}{3} \cos \frac{ny}{6}$ $(\sigma > 0)$ appear, and sketch the resulting two-dimensional patterns.

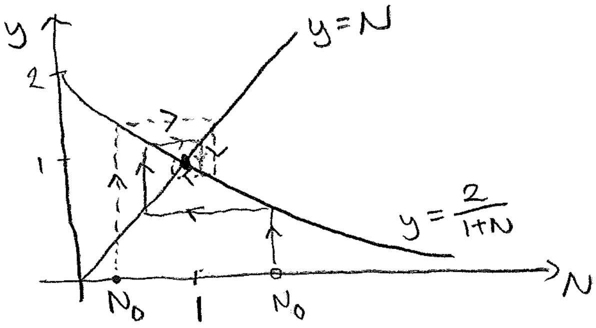
TATM38, Math. Bio. 8/1 2022, solution sketches

(1) $N_{n+1} = \frac{2}{1+N_n} = f(N_n)$. Steady states $\bar{N} = f(\bar{N}) = \frac{2}{1+\bar{N}} \Rightarrow$

$\bar{N}^2 + \bar{N} - 2 = 0 \Rightarrow \bar{N} = 1$ (or $\bar{N} = -2$ but $N_n \geq 0$), one steady state

$f'(N) = \frac{-2}{(1+N)^2} \Rightarrow |f'(\bar{N})| = |f'(1)| = |\frac{-2}{4}| = \frac{1}{2} < 1 \Rightarrow \bar{N} = 1$ stable

Cobweb (for $N \geq 0$)



$N_n \to 1, n \to \infty$ for any $N_0 > 0$

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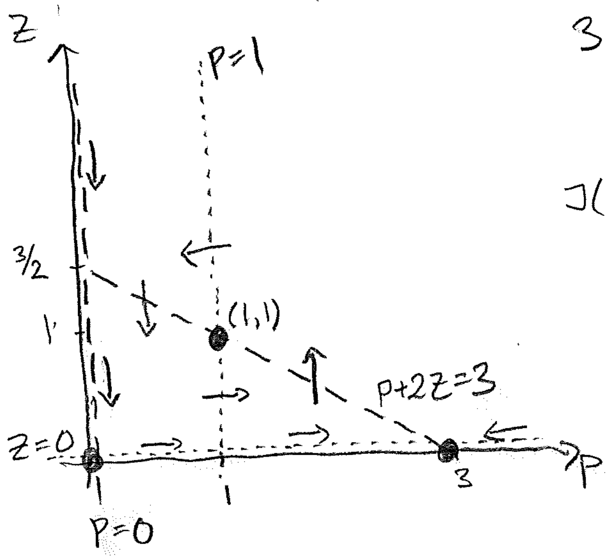
(2) $\begin{cases} P' = P(3-P-2Z) \\ Z' = Z(P-1) \end{cases}$

P nullclines: $P=0$, $P+2Z=3$, 2 lines
Z nullclines: $Z=0$, $P=1$, 2 lines

Phase plane ($P \geq 0, Z \geq 0$)



3 steady states $(\bar{P}_1, \bar{Z}_1) = (0,0)$, $(\bar{P}_2, \bar{Z}_2) = (3,0)$
$(\bar{P}_3, \bar{Z}_3) = (1,1)$

$J(P,Z) = \begin{pmatrix} 3-2P-2Z & -2P \\ Z & P-1 \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{cases} \lambda_1 = 3 > 0 \Rightarrow \text{unstable} \\ \lambda_2 = -1 < 0 \quad \text{(saddle)} \end{cases}$

$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & 2 \end{pmatrix}$ $\begin{cases} \lambda_1 = -3 < 0 \\ \lambda_2 = 2 > 0 \end{cases} \Rightarrow$ unstable (saddle)

$J(1,1) = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix} = J$ $\left.\begin{matrix} \operatorname{Tr} J = -1 < 0 \\ \det J = 2 > 0 \end{matrix}\right\} \Rightarrow$ stable

[or $\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2} \notin \mathbb{R}$
$\operatorname{Re}(\lambda_{1,2}) = -\frac{1}{2} < 0 \Rightarrow$ stable (spiral)]

If $P(0) > 0$ and $Z(0) > 0$, $(P(t), Z(t)) \to (1,1)$ as $t \to \infty$
(a balanced system where both populations coexist)

③ $\begin{cases} u_{n+1} = u_n + \frac{1}{4} v_n \\ v_{n+1} = \frac{1}{2} v_n \end{cases} \Leftrightarrow \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1/4 \\ 0 & 1/2 \end{pmatrix}}_{A} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$

Eigenvalues of A $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$

Eigenvectors $\lambda_1 = 1$: $\left(\begin{array}{cc|c} 0 & 1/4 & 0 \\ 0 & -1/2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = \frac{1}{2}$: $\left(\begin{array}{cc|c} 1/2 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow$ general solution is $\begin{pmatrix} u_n \\ v_n \end{pmatrix} = c_1 \lambda_1^n \bar{v}_1 + c_2 \lambda_2^n \bar{v}_2 = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \frac{1}{2^n} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Leftrightarrow$

$\begin{cases} u_n = c_1 + \frac{c_2}{2^n} \\ v_n = \frac{-2c_2}{2^n} \end{cases}$ (*) (and $w_n = 1 - u_n - v_n$) $\quad v_n = \frac{-2c_2}{2^n} \to 0, n \to \infty$ (for any c_2)

[that $v_n \to 0$ can also be deduced directly from $v_{n+1} = \frac{1}{2} v_n \Rightarrow v_n = k \cdot \left(\frac{1}{2}\right)^n \to 0$, $n \to \infty$]

$u_0 = 0.5$, $v_0 = 0.2 \Rightarrow$ [n=0 in (*)]

$\Rightarrow \begin{cases} 0.5 = c_1 + c_2 \\ 0.2 = -2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 0.6 \\ c_2 = -0.1 \end{cases} \Rightarrow \begin{cases} u_n = 0.6 - \frac{0.1}{2^n} \\ v_n = \frac{0.2}{2^n} \\ w_n = 1 - u_n - v_n = 0.4 - \frac{0.1}{2^n} \end{cases}$

and $\begin{cases} u_n \to 0.6 \\ w_n \to 0.4 \end{cases}$, $n \to \infty$

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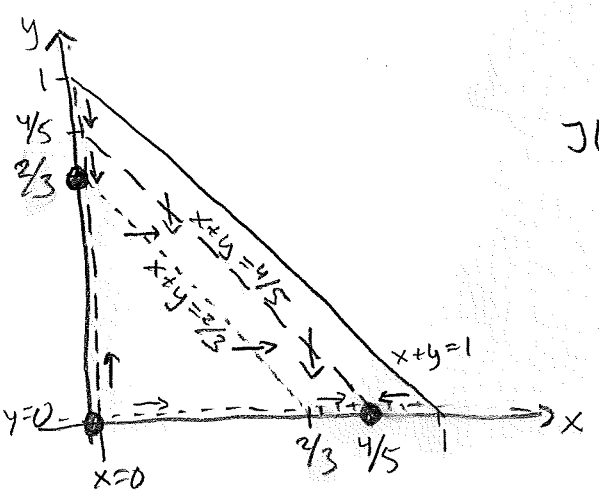
④ $\begin{cases} x' = x\left(5 - \frac{6}{2-x-y}\right) \\ y' = y\left(3 - \frac{4}{2-x-y}\right) \end{cases}$ $(x+y \le 1)$

x nullclines: $x = 0$, $5 = \frac{6}{2-x-y} \Leftrightarrow x + y = \frac{4}{5}$ 2 lines

y nullclines: $y = 0$, $3 = \frac{4}{2-x-y} \Leftrightarrow x + y = \frac{2}{3}$ 2 lines

Phase plane $(x \ge 0, y \ge 0, x + y \le 1)$



3 steady states $(\bar{x}_1, \bar{y}_1) = (0,0)$, $(\bar{x}_2, \bar{y}_2) = (\frac{4}{5}, 0)$, $(\bar{x}_3, \bar{y}_3) = (0, 2/3)$

$$J(x,y) = \begin{pmatrix} 5 - \frac{6}{2-x-y} - x \cdot \frac{6}{(2-x-y)^2} & \frac{-6x}{(2-x-y)^2} \\ \frac{-4y}{(2-x-y)^2} & 3 - \frac{4}{2-x-y} - y \frac{4}{(2-x-y)^2} \end{pmatrix}$$

$\Rightarrow J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{cases} \lambda_1 = 2 > 0 \\ \lambda_2 = 1 > 0 \end{cases}$ $\Rightarrow$ unstable

$J(\frac{4}{5}, 0) = \begin{pmatrix} -10/3 & -10/3 \\ 0 & -1/3 \end{pmatrix}$ $\begin{cases} \lambda_1 = -\frac{10}{3} < 0 \\ \lambda_2 = -\frac{1}{3} < 0 \end{cases}$ $\Rightarrow$ stable

$J(0, \frac{2}{3}) = \begin{pmatrix} 1/2 & 0 \\ -3/2 & -3/2 \end{pmatrix}$ $\begin{cases} \lambda_1 = 1/2 > 0 \\ \lambda_2 = -3/2 < 0 \end{cases}$ $\Rightarrow$ unstable (saddle)

The only stable steady state is $(\bar{x}_2, \bar{y}_2) = (\frac{4}{5}, 0)$. This means that $(x,y) \to (\frac{4}{5}, 0)$ as $t \to \infty$ if $x(0) > 0$.

Only the population $x(t)$ survives $(y(t) \to 0)$.

(5) $u(t,x) = v(t,x) + Ax + B \Rightarrow u_t = v_t,\ u_x = v_x + A,\ u_{xx} = v_{xx}$ so

$$\begin{cases} 3u_t = u_{xx} \Leftrightarrow 3v_t = v_{xx} \\ u(t,0) = v(t,0) + A\cdot 0 + B = 1 \\ u(t,1) = v(t,1) = A\cdot 1 + B = 0 \\ u(0,x) = v(0,x) + Ax + B = 1 - x + \sin(\pi x) + 2\sin(3\pi x) \end{cases}$$

Want $v(t,0) = v(t,1) = 0 \Rightarrow \underline{B = 1,\ A = -1}$

$\Downarrow$

$v(0,x) = u(0,x) - Ax - B =$
$= u(0,x) + x - 1 =$
$= \sin(\pi x) + 2\sin(3\pi x)$

IBVP for $v(t,x)$

$$\begin{cases} 3v_t = v_{xx} \\ v(t,0) = 0 \\ v(t,1) = 0 \\ v(0,x) = \sin(\pi x) + 2\sin(3\pi x) \end{cases}$$

seperation of variables:
$v(t,x) = T(t)X(x) \Rightarrow \dfrac{3T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = \lambda = \text{constant}$

$\Rightarrow T(t) = e^{\lambda t/3}$

$v(t,0) = v(t,1) = 0 \Rightarrow X(0) = X(1) = 0$

$$\left.\begin{aligned} X''(x) - \lambda X(x) = 0 \\ X(0) = X(1) = 0 \end{aligned}\right\} \Rightarrow X_n(x) = \sin(n\pi x),\ n = 1,2,3,\dots,\ \lambda = -n^2\pi^2$$

$$\Rightarrow v(t,x) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 t/3} \sin(n\pi x)$$

$$v(0,x) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) = \sin(\pi x) + 2\sin(3\pi x) \Rightarrow \alpha_1 = 1, \alpha_3 = 2,\ \alpha_n = 0,\ n \neq 1,3$$

$$\Rightarrow v(t,x) = e^{-\pi^2 t/3}\sin \pi x + 2e^{-9\pi^2 t/3}\sin(3\pi x) \Rightarrow$$

$$\underline{u(t,x)} = v(t,x) - x + 1 = \underline{1 - x + e^{-\pi^2 t/3}\sin \pi x + 2e^{-3\pi^2 t}\sin 3\pi x}$$

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(6) With $A = J(\bar{u},\bar{v}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ the conditions for Turing instability are

1. $\operatorname{Tr} A < 0$
2. $\det A > 0$
3. $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A}$

$J_1 = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}$ has $\det J_2 = -2 \Rightarrow$ 2. not satisfied

$J_3 = \begin{pmatrix} -1 & 3 \\ -1 & -2 \end{pmatrix}$ has $\operatorname{Tr} J_3 = -3 < 0$ and $\det J_3 = 5 > 0$ but $a_{11} = -1 < 0$ and $a_{22} = -2 < 0 \Rightarrow$ 3. cannot be satisfied ($D_1 > 0, D_2 > 0$)

$J_2 = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$ has $\operatorname{Tr} J_2 = -1 < 0$, $\det J_2 = 1 > 0$ and $a_{11} = 1 > 0 \Rightarrow$ 1. and 2. satisfied, and 3. can be satisfied (dep. on values of D_1 and D_2)

Study $A = J_2$ with $D_1 = 1, D_2 = 8$

$$\left.\begin{aligned} a_{11}D_2 + a_{22}D_1 = 1\cdot 8 - 2\cdot 1 = 6 \\ 2\sqrt{D_1D_2\det A} = 2\sqrt{8} = 4\sqrt{2} < 6 \end{aligned}\right\} \Rightarrow \text{3. satisfied}$$

$$P = \frac{a_{11}D_2 + a_{22}D_1}{D_1 D_2} = \frac{6}{8} = \frac{3}{4},\ W = \frac{\det A}{D_1 D_2} = \frac{1}{8},\ \Delta = \sqrt{\frac{P^2}{4} - W} = \sqrt{\frac{9}{64} - \frac{8}{64}} = \frac{1}{8} \Rightarrow$$

Unstable perturbations appear for $\frac{P}{2} - \Delta < Q^2 < \frac{P}{2} + \Delta \Leftrightarrow \frac{1}{4} < Q^2 < \frac{1}{2}$

where $Q^2 = \dfrac{m^2}{9} + \dfrac{n^2}{36} \Rightarrow \frac{1}{4} < \frac{m^2}{9} + \frac{n^2}{36} < \frac{1}{2} \Leftrightarrow 9 < 4m^2 + n^2 < 18$

(cont.)

⑥ cont. $9 < 4m^2 + n^2 < 18$ find integer solutions (m, n):

$(0,4), (1,3), (2,0), (2,1)$ are the four possibilities. Patterns: