Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2022-10-29 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. A population $N(t)$ is described by a logistic equation with a linear fishing term kN:

$$\frac{dN}{dt} = r(1 - \frac{N}{B})N - kN$$

Here $r > 0$, $B > 0$ and $k > 0$ are constants.
(a) Assume $k < r$. Find all steady states (= equilibrium points) and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$ if $N(0) > 0$?
(b) Assume now $k > r$. Find all steady states and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$?
Interpret the differences between the cases $k < r$ and $k > r$.

2. A population is divided into two age classes. The number of individuals at time n is a_n in the youngest class and b_n in the oldest class. The survival rate from the youngest class to the oldest is 80%. The average number of births from individuals in the two classes are 0.8 and 0.6, respectively. This gives the linear system

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.6 \\ 0.8 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} .$$

Find the general solution to the system. For large n, by how much do a_n and b_n approximately increase each time step, and what is the approximate age distribution of the population (that is, what fractions of the entire population are in the different age classes a_n and b_n)?

3. A model for two mutualist populations $x(t)$ and $y(t)$ is given by

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x + y) \\ \dfrac{dy}{dt} = y(1 - 2y + x) \end{cases}$$

(Mutualist means that both populations benefit from the presence of the other population, which is shown by the plus sign at the xy-terms in both equations.)
Find all steady states and determine their stability. Draw, for $x \geq 0$ and $y \geq 0$, a phase plane picture (with nullclines and directions of the vector field). What happens to the populations as $t \to \infty$ if $x(0) > 0$ and $y(0) > 0$?

PLEASE TURN

4. Let $S(t)$, $I(t)$ and $R(t)$ be the numbers of susceptibles, infective, and removed, respectively, in a SIRS epidemic model. The total population N is constant, $N = 200$, and with $R(t) = N - S(t) - I(t)$ it is sufficient to study a two-dimensional system for $S(t)$ and $I(t)$. With certain choices of parameter values, the system is

$$\begin{cases} \dfrac{dS}{dt} = \dfrac{200 - S - I}{6} - \dfrac{SI}{150} \\ \dfrac{dI}{dt} = \dfrac{SI}{150} - \dfrac{2I}{3} \end{cases}$$

Find all steady states of the system and determine their stability. Draw a phase plane picture (with nullclines and directions of the vector field). Note that $S \geq 0$, $I \geq 0$, and $S+I \leq 200$. What happens to the number susceptibles and infective as $t \to \infty$ if $I(0) > 0$?

5. For $0 < x < 1$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} u_t = 2u_{xx} - 3u \\ u(t, 0) = 0 \\ u(t, 1) = 0 \\ u(0, x) = \pi \end{cases}$$

Hint: put $u(t, x) = v(t, x)e^{\alpha t}$.

6. Consider a Schnakenberg two-component reaction-diffusion system in two space dimensions with concentrations $u(t, x, y)$ and $v(t, x, y)$:

$$\begin{cases} u_t = u^2v - u + \frac{11}{36} + D_1\left(u_{xx} + u_{yy}\right) \\ v_t = \frac{25}{36} - u^2v + D_2\left(v_{xx} + v_{yy}\right) \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability. Suppose now that $D_1 = 1/6$, $D_2 = 6$, that $0 < x < L_1 = 3\pi$, $0 < y < L_2 = \sqrt{3}\pi$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}$. For what values of (m, n) can we have diffusive instabilities ($\sigma > 0$) ? Sketch the resulting two-dimensional patterns.

TATM38, Math. Bio., 29/10 2022, solution sketches

(1) $\frac{dN}{dt} = f(N) = r(1-\frac{N}{B})N - kN = N(r-k-\frac{rN}{B})$

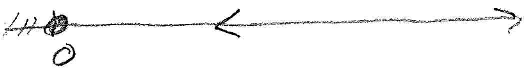
Steady states: $f(\bar{N})=0 \Rightarrow \bar{N}_1 = 0,\ \bar{N}_2 = \frac{B}{r}(r-k) > 0$ only if $k<r$.

Stable if $f'(\bar{N}_j) < 0$. $f'(N) = r-k-\frac{2rN}{B} \Rightarrow f'(\bar{N}_1) = r-k,\ f'(\bar{N}_2) = k-r$

a) $k<r \Rightarrow$ 2 steady states $(\bar{N}_2>0)$, $f'(\bar{N}_1)>0$ unstable, $f'(\bar{N}_2)<0$ stable

$N(t) \to \bar{N}_2$ if $N(0)>0$, $t\to\infty$

b) $k>r \Rightarrow \bar{N}_2 < 0$ and $f'(\bar{N}_1)<0$ stable



$N(t) \to \bar{N}_1 = 0 \quad \forall N(0) \geq 0,\ t\to\infty$

With $k>r$ there is too much fishing so the population dies out $(N(t)\to 0)$.
With $k<r$ a balanced population can be kept.

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(2) $\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = A \begin{pmatrix} a_n \\ b_n \end{pmatrix}$, $\det(A-\lambda I) = \begin{vmatrix} 0.8-\lambda & 0.6 \\ 0.8 & -\lambda \end{vmatrix} = \lambda^2 - 0.8\lambda - 0.48 = 0 \Rightarrow$

$\lambda = 0.4 \pm \sqrt{0.16+0.48} = 0.4 \pm 0.8$. Eigenvectors

$\lambda_1 = 1.2: \left(\begin{array}{cc|c} -0.4 & 0.6 & 0 \\ 0.8 & -1.2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda_2 = -0.4: \left(\begin{array}{cc|c} 1.2 & 0.6 & 0 \\ 0.8 & 0.4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow$

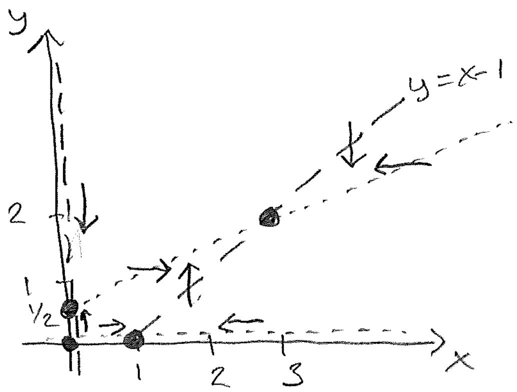
General solution is $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = c_1 \cdot (1.2)^n \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 (-0.4)^n \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$(1.2)^n$ grows and $(-0.4)^n \to 0$ for large $n \Rightarrow$

$\begin{pmatrix} a_n \\ b_n \end{pmatrix} \approx c_1 \cdot (1.2)^n \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ for large n so both a_n and b_n increase by approx. 20% each time step, and $\frac{3}{5}$ of the total population will be in class a_n, and $\frac{2}{5}$ in class b_n.

[of course, exponential growth cannot go on forever in real life]

(3) $\begin{cases} x' = x(1-x+y) \\ y' = y(1-2y+x) \end{cases}$ x nullclines $x=0$ and $y=x-1$
y nullclines $y=0$ and $y=\frac{x+1}{2}$



4 steady states: $(\bar{x}_1,\bar{y}_1)=(1,0)$, $(\bar{x}_2,\bar{y}_2)=(0,1/2)$, $(\bar{x}_3,\bar{y}_3)=(3,2)$, $(\bar{x}_4,\bar{y}_4)=(0,0)$

$$J(x,y)=\begin{pmatrix} 1-2x+y & x \\ y & 1-4y+x \end{pmatrix} \Rightarrow$$

$$J(1,0)=\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \quad \left.\begin{matrix} \lambda_1=2>0 \\ \lambda_2=-1<0 \end{matrix}\right\} \Rightarrow \text{unstable (saddle)}$$

$$J(0,\tfrac{1}{2})=\begin{pmatrix} 3/2 & 0 \\ 1/2 & -1 \end{pmatrix} \quad \left.\begin{matrix} \lambda_1=3/2>0 \\ \lambda_2=-1<0 \end{matrix}\right\} \Rightarrow \text{unstable (saddle)}, \quad J(0,0)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \lambda_1=\lambda_2=1>0 \Rightarrow \text{unstable}$$

$$J(3,2)=\begin{pmatrix} -3 & 3 \\ 2 & -4 \end{pmatrix} \Rightarrow \left.\begin{matrix} \lambda_1=-6<0 \\ \lambda_2=-1<0 \end{matrix}\right\} \Rightarrow \text{stable (and } \lambda_{1,2} \text{ real} \Rightarrow \text{no spiral)}$$

[can also check $\left.\begin{matrix} \operatorname{Tr} J=-7<0 \\ \det J=6>0 \end{matrix}\right\} \Rightarrow$ stable]

$(x(t),y(t)) \to (3,2)$ as $t\to\infty$ if $x(0)>0, y(0)>0$ (coexistence of populations with higher levels than the levels 1 and 1/2 the populations would have if they were alone)

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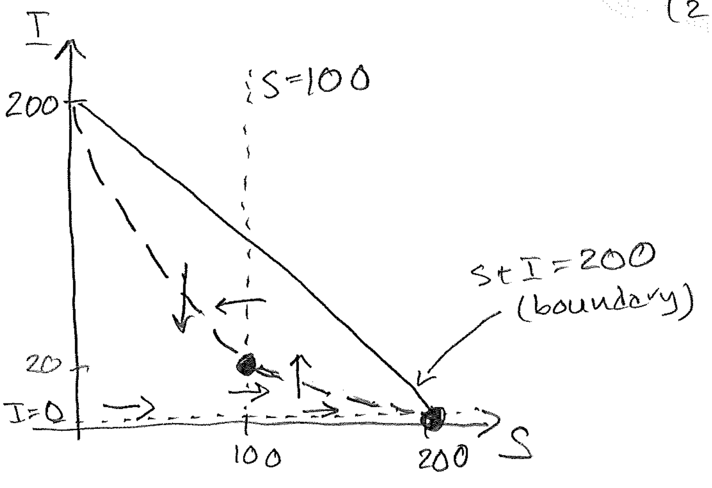
(4) $\begin{cases} S' = \frac{1}{6}(200-S-I)-\frac{1}{150}SI \\ I' = I\left(\frac{S}{150}-\frac{2}{3}\right) \end{cases}$ S nullcline $SI=25(200-S-I) \Rightarrow I(25+S)=25(200-S)$

I nullclines $I=0$, $S=100$ (2 lines)

$$\Rightarrow \underbrace{I=\frac{25(200-S)}{25+S}}_{\text{curve throug } (200,0),\ (0,200) \text{ and } (100,20)} \quad (*)$$



Two steady states:
$(\bar{S}_1,\bar{I}_1)=(200,0)$, $(\bar{S}_2,\bar{I}_2)=(100,20)$

$$J(S,I)=\begin{pmatrix} -\frac{1}{6}-\frac{I}{150} & -\frac{1}{6}-\frac{S}{150} \\ \frac{I}{150} & \frac{S}{150}-\frac{2}{3} \end{pmatrix}$$

$$J(200,0)=\begin{pmatrix} -1/6 & -3/2 \\ 0 & 2/3 \end{pmatrix} \quad \left.\begin{matrix} \lambda_1=-1/6<0 \\ \lambda_2=2/3>0 \end{matrix}\right\} \Rightarrow \text{unstable (saddle)}$$

$$J(100,20)=\begin{pmatrix} -3/10 & -5/6 \\ 2/15 & 0 \end{pmatrix}=J_2 \quad \left.\begin{matrix} \operatorname{Tr} J_2=-\frac{3}{10}<0 \\ \det J_2=\frac{1}{9}>0 \end{matrix}\right\} \Rightarrow \text{stable}$$

[$(\operatorname{Tr} J_2)^2-4\det J_2=\frac{9}{100}-\frac{4}{9}<0$
$\Rightarrow$ complex eigenvalues
$\Rightarrow$ spiraling approach]

$(S(t),I(t)) \to (100,20)$, $t\to\infty$ if $I(0)>0$

(so $R(t)\to 80$; with $\bar{I}=20$ disease is established)

(5) $u(t,x)=v(t,x)e^{\alpha t} \Rightarrow u_t=(v_t+\alpha v)e^{\alpha t},\ u_{xx}=v_{xx}e^{\alpha t} \Rightarrow$

$u_t=2u_{xx}-3u \iff (v_t+\alpha v)e^{\alpha t}=(2v_{xx}-3v)e^{\alpha t}$, Take $\alpha=-3 \Rightarrow v_t=2v_{xx}$

IBVP for $v(t,x)$:

$$\begin{cases} v_t=2v_{xx} \\ v(t,0)=u(t,0)e^{3t}=0 \\ v(t,1)=u(t,1)e^{3t}=0 \\ v(0,x)=u(0,x)e^{3\cdot 0}=\pi \end{cases}$$

Separation of variables: $v(t,x)=T(t)X(x) \Rightarrow$

$$\frac{T'(t)}{2T(t)}=\frac{X''(x)}{X(x)}=\lambda=\text{constant} \Rightarrow T(t)=e^{2\lambda t}$$

$v(t,0)=v(t,1)=0 \Rightarrow X(0)=X(1)=0$

$$\begin{cases} X''(x)-\lambda X(x)=0 \\ X(0)=X(1)=0 \end{cases} \Rightarrow X_n(x)=\sin(n\pi x),\ n=1,2,\dots,\ \lambda=-n^2\pi^2 \Rightarrow$$

$$v(t,x)=\sum_{n=1}^{\infty} b_n e^{-2n^2\pi^2 t}\sin(n\pi x) \Rightarrow v(0,x)=\sum_{n=1}^{\infty} b_n\sin(n\pi x)=\pi$$ sin-series of $f(x)=\pi$ on $0<x<1$

$$\Rightarrow b_n=\frac{2}{1}\int_0^1 \pi\sin(n\pi x)\,dx=2\left[-\frac{\cos(n\pi x)}{n}\right]_0^1=\frac{2}{n}\left(-(-1)^n+1\right)=\begin{cases} 4/n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\underline{u(t,x)}=v(t,x)e^{-3t}=\underline{2e^{-3t}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}e^{-2n^2\pi^2 t}\sin(n\pi x)}$$

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(6) $$\begin{cases} u_t=u^2v-u+\frac{11}{36}+D_1(u_{xx}+u_{yy}) \\ v_t=\frac{25}{36}-u^2v+D_2(v_{xx}+v_{yy}) \end{cases}$$

Spatially uniform steady state $(\bar u,\bar v)$

$$\begin{cases} \bar u^2\bar v-\bar u+\frac{11}{36}=0 \\ \frac{25}{36}-\bar u^2\bar v=0 \end{cases}$$ add $\Rightarrow 1-\bar u=0 \Rightarrow \underline{\bar u=1}$

$\Rightarrow \underline{\bar v=\frac{25}{36}}$

Stability for $D_1=D_2=0$

$$J(u,v)=\begin{pmatrix} 2uv-1 & u^2 \\ -2uv & -u^2 \end{pmatrix} \Rightarrow J(1,\tfrac{25}{36})=\begin{pmatrix} \frac{7}{18} & 1 \\ -\frac{25}{18} & -1 \end{pmatrix}=J$$

$\left.\begin{array}{l} \mathrm{Tr}\,J=-\frac{11}{18}<0 \\ \det J=1>0 \end{array}\right\} \Rightarrow$ <u>stable</u>

Turing condition $J_{11}D_2+J_{22}D_1>2\sqrt{D_1D_2\det J} \iff \underline{\frac{7}{18}D_2-D_1>2\sqrt{D_1D_2}}$

$D_1=\frac{1}{6}, D_2=6$ gives $\frac{42}{18}-\frac{1}{6}=\frac{13}{6}>2=2\sqrt{\frac{1}{6}\cdot 6\cdot 1}$ so Turing condition satisfied

Unstable perturbations

$$0>\det(J-Q^2D)=\begin{vmatrix} \frac{7}{18}-\frac{Q^2}{6} & 1 \\ -\frac{25}{18} & -1-6Q^2 \end{vmatrix}=Q^4-\frac{13}{6}Q^2+1=\left(Q^2-\frac{13}{12}\right)^2+\overbrace{1-\frac{169}{144}}^{-25/144}=\left(Q^2-\frac{13}{12}\right)^2-\left(\frac{5}{12}\right)^2$$

$$\iff \left|Q^2-\frac{13}{12}\right|<\frac{5}{12} \iff -\frac{5}{12}<Q^2-\frac{13}{12}<\frac{5}{12} \iff \frac{8}{12}<Q^2<\frac{18}{12} \iff \frac{2}{3}<Q^2<\frac{3}{2}$$

$Q^2=\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\pi^2=\frac{m^2}{9}+\frac{n^2}{3}$ so $\frac{2}{3}<\frac{m^2}{9}+\frac{n^2}{3}<\frac{3}{2} \iff 6<m^2+3n^2<\frac{27}{2}=13{,}5$

Possible integer solutions $(m,n)=(3,0),(2,1),(3,1),(0,2),(1,2)$

Patterns: