Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2023-01-05 , 14.00 - 19.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. The interaction between two species $x(t)$ and $y(t)$ is modeled by the linear system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

Determine the general solution to this system and draw a picture of the entire phase plane (including negative x and y). Is the steady state (= equilibrium point) $(0, 0)$ stable?
If the initial values are $x(0) = 3$ and $y(0) = 1$, what are $x(1)$ and $y(1)$?

2. The population $0 \leq N_n \leq 2\pi$ in a time discrete population model satisfies

$$N_{n+1} = N_n + \sin N_n \quad , \quad n = 0, 1, 2, \dots$$

Find the steady states (= equilibrium points) of the model and determine their stability. Sketch a cobweb diagram for some N_0. What happens to the population as $n \to \infty$ for different N_0, where $0 \leq N_0 \leq 2\pi$?

3. Let $x(t)$ and $y(t)$ be prey and predator populations, respectively, in a model of logistic type for both populations, in which the carrying capacity for predators is modeled by the number of prey:

$$\begin{cases} \dfrac{dx}{dt} = x(2 - x - y) \\ \dfrac{dy}{dt} = y(1 - \dfrac{y}{x}) \end{cases}$$

Note that we cannot have $x = 0$ in the second equation so we assume that $x > 0$ and $y \geq 0$. Find the steady states (= equilibrium points) and determine their stability. Draw a phase plane picture (with nullclines and directions of the vector field) for $x > 0$ and $y \geq 0$. If $x(0) > 0$ and $y(0) > 0$, what happens to the populations as $t \to \infty$?

PLEASE TURN

4. Let S_n and I_n be the numbers of susceptibles and infective, respectively, in a time discrete disease model with vaccination:

$$\begin{cases} S_{n+1} = & (1-p)S_n - \frac{1}{2}S_n I_n + 1 \\ I_{n+1} = & \frac{1}{2}S_n I_n \end{cases} \quad , \; n = 0, 1, 2, \ldots$$

Here p is the fraction of susceptibles getting vaccinated.
Consider the two cases $0 < p < \frac{1}{2}$ (low degree of vaccination), and $\frac{1}{2} < p < 1$ (high degree of vaccination). In both cases, find all steady states of the system and determine their stability. Give an interpretation of the difference in result between the two cases.

5. For $0 < x < \pi$, $0 < y < \pi$, and $t > 0$, solve the initial-boundary value problem (IBVP) in two space-dimensions for $u(t, x, y)$

$$\begin{cases} u_t = u_{xx} + u_{yy} \\ u(t, 0, y) = u(t, \pi, y) = 0 \\ u_y(t, x, 0) = u_y(t, x, \pi) = 0 \\ u(0, x, y) = 3\sin 2x \cos y - \sin x \cos 3y \end{cases}$$

6. Consider a glycolyctic oscillator with diffusion in one space-dimension with concentrations $f(x, t)$ and $g(x, t)$:

$$\begin{cases} f_t = 1 - 2f - fg^2 + D_1 \, f_{xx} \\ g_t = 2f + fg^2 - g + D_2 \, g_{xx} \end{cases}$$

Find the spatially uniform steady state $(\bar{f}, \bar{g})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
Show that Turing diffusive instability is impossible in this model, so perturbations cannot give rise to spatial patterns in the concentrations.

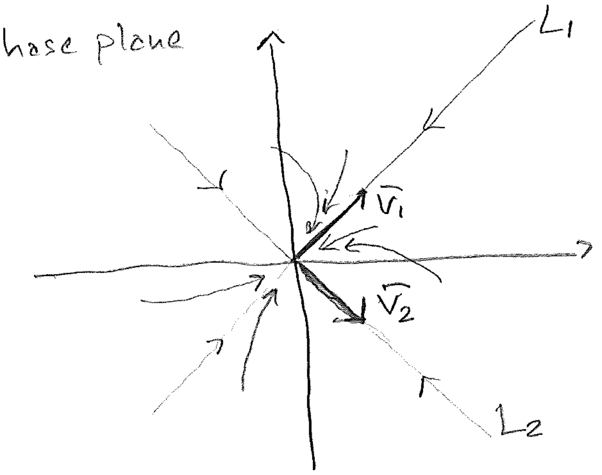
TATM38, Math.Bio., 5/1 2023, solution sketches

(1) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ Eigenvalues $\begin{cases} \lambda_1 = -1 < 0 \\ \lambda_2 = -3 < 0 \end{cases} \Rightarrow (0,0)$ stable steady state

Eigenvectors $\lambda_1 = -1 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -3 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Rightarrow$ general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}, t \to \infty$$

Phase plane



$e^{-3t} \to 0$ faster than e^{-t} as $t \to \infty$
$\Rightarrow$ solutions approach L_1 for large t (unless they started on L_2)

$$t=0: \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = 2e^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ so}$$

$$x(1) = 2e^{-1} + e^{-3},\ y(1) = 2e^{-1} - e^{-3}$$

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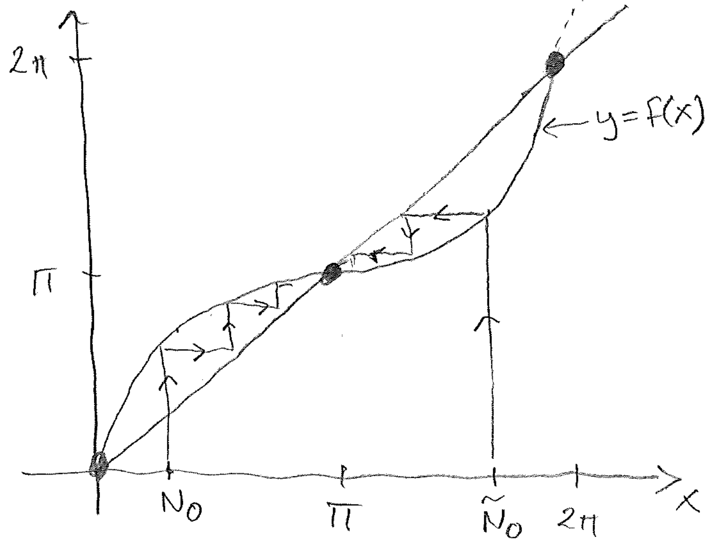
(2) $N_{n+1} = N_n + \sin N_n = f(N_n)$. Steady states $f(\bar{N}) = \bar{N} \iff$
$\bar{N} + \sin\bar{N} = \bar{N} \iff \sin\bar{N} = 0 \Rightarrow \bar{N}_1 = 0, \bar{N}_2 = \pi, \bar{N}_3 = 2\pi \quad (0 \le \bar{N} \le 2\pi)$
3 steady states for $0 \le \bar{N} \le 2\pi$. Stable if $|f'(\bar{N}_j)| < 1$, $f'(N) = 1 + \cos N \Rightarrow$

$|f'(0)| = |1 + \cos 0| = 2 > 1 \Rightarrow \bar{N}_1 = 0$ unstable
$|f'(\pi)| = |1 + \cos\pi| = 0 < 1 \Rightarrow \bar{N}_2 = \pi$ stable
$|f'(2\pi)| = |1 + \cos 2\pi| = 2 > 1 \Rightarrow \bar{N}_3 = 2\pi$ unstable

Cobweb for $0 \le x \le 2\pi$



$f'(x) = 1 + \cos x \ge 0\ \forall x$ ($\Rightarrow$ increasing)
$f'(x) = 0$ only at $x = \pi$ if $0 \le x \le 2\pi$
(and $0 \le f(x) \le 2\pi$ for $0 \le x \le 2\pi$)

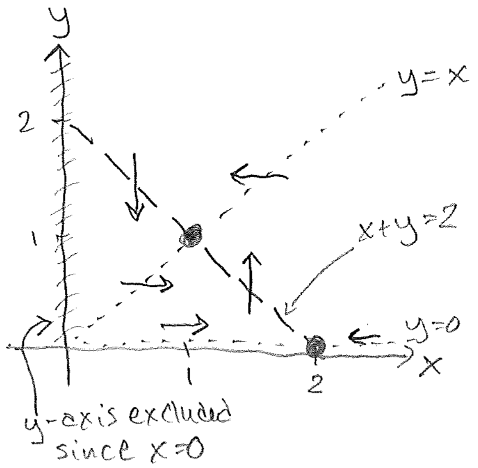
We get

$$N_n \to \begin{cases} 0 & \text{if } N_0 = 0 \\ \pi & \text{if } 0 < N_0 < 2\pi \\ 2\pi & \text{if } N_0 = 2\pi \end{cases}$$

as $n \to \infty$

③ $\begin{cases} x' = x(2-x-y) \\ y' = y\left(1-\frac{y}{x}\right) \end{cases}$ x-nullclines $x+y=2$ ($x=0$ excluded because $x>0$ required in $\frac{y}{x}$)

y-nullclines $y=0$, $y=x$



2 steady states $(\bar{x}_1,\bar{y}_1)=(1,1)$, $(\bar{x}_2,\bar{y}_2)=(2,0)$

$$J(x,y) = \begin{pmatrix} 2-2x-y & -x \\ y^2/x^2 & 1-2y/x \end{pmatrix} \Rightarrow$$

$$J(1,1) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},\ \lambda_{1,2} = -1 \pm i \Rightarrow Re(\lambda_{1,2}) = -1 < 0$$
$\Rightarrow$ stable (spiral)

[or $Tr\, J(1,1) = -2 < 0$, $\det J(1,1) = 2 > 0 \Rightarrow$ stable]

$$J(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = -2 < 0 \\ \lambda_2 = 1 > 0 \end{cases} \Rightarrow \text{unstable (saddle)}$$

$(x(t),y(t)) \to (1,1)$, $t\to\infty$ if $x(0)>0$ and $y(0)>0$ (if $y(0)=0$, $(x,y)\to(2,0)$, $t\to\infty$)

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④ $\begin{cases} S_{n+1} = (1-p)S_n - \frac{1}{2}S_n I_n + 1 \\ I_{n+1} = \frac{1}{2}S_n I_n \end{cases}$ steady states $\begin{cases} \bar{S} = S_{n+1} = S_n \\ \bar{I} = I_{n+1} = I_n \end{cases} \Rightarrow$

$$\begin{cases} \bar{S} = (1-p)\bar{S} - \frac{1}{2}\bar{S}\bar{I} + 1 \\ \bar{I} = \frac{1}{2}\bar{S}\bar{I} \end{cases} \Leftrightarrow \begin{cases} 0 = -p\bar{S} - \frac{1}{2}\bar{S}\bar{I} + 1 \quad (1) \\ \bar{I}\left(1-\frac{1}{2}\bar{S}\right) = 0 \Rightarrow \bar{I}=0 \text{ or } \bar{S}=2 \end{cases}$$

$\bar{I}=0$ in (1) $\Rightarrow \bar{S} = \frac{1}{p}$, $\bar{S}=2$ in (1) $\Rightarrow \bar{I} = 1-2p$ $\begin{cases} >0 \text{ if } p<1/2 \\ <0 \text{ if } p>1/2 \end{cases}$

<u>If $0<p<\frac{1}{2}$</u>, two steady states $(\bar{S}_1,\bar{I}_1) = (\frac{1}{p},0)$, $(\bar{S}_2,\bar{I}_2) = (2,1-2p)$

$$J(S,I) = \begin{pmatrix} 1-p-\frac{1}{2}I & -\frac{1}{2}S \\ \frac{1}{2}I & \frac{1}{2}S \end{pmatrix} \Rightarrow J(\tfrac{1}{p},0) = \begin{pmatrix} 1-p & -\frac{1}{2p} \\ 0 & \frac{1}{2p} \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 1-p \\ \lambda_2 = \frac{1}{2p} \end{cases} \Rightarrow$$

(*) $\begin{matrix} |\lambda_1| = 1-p < 1 \\ |\lambda_2| = \frac{1}{2p} > 1 \text{ since } p<\frac{1}{2} \end{matrix} \Rightarrow (\frac{1}{p},0)$ unstable, $J(2,1-2p) = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2}-p & 1 \end{pmatrix} = J_2$ $\quad Tr\, J_2 = \frac{3}{2}$, $\det J_2 = 1-p$

Jury test $\underbrace{|Tr\, J_2|}_{3/2} < \underbrace{1+\det J_2}_{2-p} < 2 \Rightarrow$ stable

satisfied since $p<\frac{1}{2}$ satisfied ($p>0$)

<u>If $\frac{1}{2}<p<1$</u>, one steady state $(\bar{S}_1,\bar{I}_1) = (\frac{1}{p},0)$

From (*) $\begin{cases} |\lambda_1| = 1-p < 1 \\ |\lambda_2| = \frac{1}{2p} < 1 \text{ since } p>\frac{1}{2} \end{cases} \Rightarrow$ <u>stable</u>

Interpretation: with low p ($p<\frac{1}{2}$), $(\bar{S}_2,\bar{I}_2) = (2,1-2p)$ is stable which indicates that the disease will stay in the population with $I = 1-2p$ infective. With high p ($p>\frac{1}{2}$), $(\bar{S}_1,\bar{I}_1) = (\frac{1}{p},0)$ is stable so the disease disappears ($I=0$).

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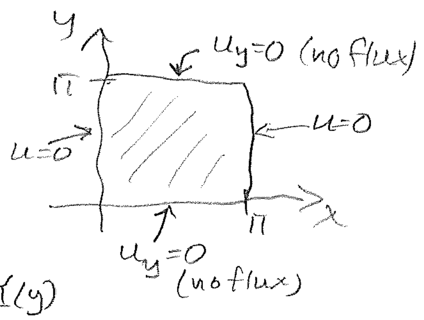
(5) $\begin{cases} u_t = u_{xx} + u_{yy} & (1) \\ u(t,0,y) = u(t,\pi,y) = 0 & \\ u_y(t,x,0) = u_y(t,x,\pi) = 0 & \end{cases}$ boundary conditions (2a) (2b)

$u(0,x,y) = 3\sin 2x \cos y - \sin x \cos 3y$ initial condition (3)



Separation of variables $u(t,x,y) = T(t)X(x)Y(y)$

$$\underset{(1)}{\Rightarrow} \frac{T'}{T} = \underbrace{\frac{X''}{X}}_{=\lambda} + \underbrace{\frac{Y''}{Y}}_{=\mu}, \quad \lambda, \mu \text{ constants}$$

$(2a) \Rightarrow X(0) = X(\pi) = 0$

$(2b) \Rightarrow Y'(0) = Y'(\pi) = 0$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \Rightarrow X_n(x) = \sin nx, \; n = 1,2,\ldots, \; \lambda = -n^2$$

$$\begin{cases} Y'' - \mu Y = 0 \\ Y'(0) = Y'(\pi) = 0 \end{cases} \Rightarrow Y_m(y) = \cos my, \; m = 0,1,2,\ldots, \; \mu = -m^2$$

$$\Rightarrow T' = -(n^2+m^2)T \Rightarrow T_{n,m}(t) = e^{-(n^2+m^2)t}$$

$$\Rightarrow u(t,x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \alpha_{n,m} e^{-(n^2+m^2)t} \sin nx \cos my \quad \text{solves (1), (2a) and (2b)}$$

$$(3) \Rightarrow u(0,x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \alpha_{n,m} \sin nx \cos my = 3\sin 2x \cos y - \sin x \cos 3y. \text{ Identify}$$

$\alpha_{2,1} = 3$, $\alpha_{1,3} = -1$, other $\alpha_{n,m} = 0 \Rightarrow$

$$\underline{u(t,x,y) = 3e^{-5t}\sin 2x \cos y - e^{-10t}\sin x \cos 3y}$$

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(6) $\begin{cases} f_t = 1 - 2f - fg^2 + D_1 f_{xx} \\ g_t = 2f + fg^2 - g + D_2 g_{xx} \end{cases}$

Spatially uniform steady state $(\bar f, \bar g)$:

$$\begin{cases} 1 - 2\bar f - \bar f \bar g^2 = 0 \\ 2\bar f + \bar f \bar g^2 - \bar g = 0 \end{cases} \underset{\text{(add)}}{\Rightarrow} 1 - \bar g = 0 \Rightarrow \bar g = 1 \Rightarrow \bar f = \frac{1}{3}$$

$$\Rightarrow \underline{(\bar f, \bar g) = (\tfrac{1}{3}, 1)}$$

Stability for $D_1 = D_2 = 0$

$$J(f,g) = \begin{pmatrix} -2-g^2 & -2fg \\ 2+g^2 & 2fg-1 \end{pmatrix} \Rightarrow J(\tfrac{1}{3},1) = \begin{pmatrix} -3 & -\frac{2}{3} \\ 3 & -\frac{1}{3} \end{pmatrix} \Rightarrow \begin{cases} \mathrm{Tr}\, J = -\frac{10}{3} < 0 \\ \det J = 3 > 0 \end{cases} \Rightarrow \underline{\text{stable}}$$

Turing condition for diffusive instability

$$j_{11} D_2 + j_{22} D_1 > 2\sqrt{D_1 D_2 \det J} \iff \underbrace{-3D_2 - \tfrac{1}{3}D_1}_{<0} > \underbrace{2\sqrt{3D_1 D_2}}_{>0}$$

impossible to satisfy since $D_1 > 0$, $D_2 > 0$.

[with other coefficients in the equations, a glycolytic oscillator can have diffusive instab.]