Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2023-08-21 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. A population $N(t)$ is described by a logistic equation with a cubic fishing term:

$$\frac{dN}{dt} = 2(1 - \frac{N}{6})N - \frac{N^3}{3}$$

Find all steady states (= equilibrium points) and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$ if $N(0) > 0$?

2. A model for the mixing of two populations $n(t)$ and $p(t)$ is given by

$$\begin{cases} \dfrac{dn}{dt} = 5 - 2n + p \\ \dfrac{dp}{dt} = 1 + 2n - 3p \end{cases}$$

Find all steady states (= equilibrium points) and determine their stability. Draw a phase plane picture (with nullclines and directions of the vector field) for $n \geq 0$ and $p \geq 0$. What happens to the populations as $t \to \infty$?
Find a second-order (non-homogeneous) ordinary differential equation for $n(t)$ and solve it explicitly. Determine the limit of $n(t)$ as $t \to \infty$ also from this expression.

3. In a time discrete model two populations x_n and y_n compete for the same resources, and their time evolution is given by

$$\begin{cases} x_{n+1} = x_n(2 - \dfrac{x_n}{2} - \dfrac{y_n}{2}) \\ y_{n+1} = y_n(2 - \dfrac{x_n}{4} - \dfrac{3y_n}{4}) \end{cases}$$

Find all steady states of the system and determine their stability.

PLEASE TURN

4. Let $x(t)$ and $y(t)$ be prey and predator populations, respectively, decribed by a Lotka-Volterra predator-prey model with modified logistic prey growth:

$$\begin{cases} \dfrac{dx}{dt} = x(1-x^2) - xy \\ \dfrac{dy}{dt} = -y + 2xy \end{cases}$$

Find all steady states and determine their stability. Draw, for $x \geq 0$ and $y \geq 0$, a phase plane picture (with nullclines and directions of the vector field). What happens to the populations as $t \to \infty$ if $x(0) > 0$ and $y(0) > 0$?

5. For $0 < x < 2\pi$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t,x)$

$$\begin{cases} u_t = 3u_{xx} - 6u_x \\ u(t,0) = 0 \\ u(t,2\pi) = 0 \\ u(0,x) = e^x(2\sin\frac{x}{2} - \sin\frac{3x}{2}) \end{cases}$$

Hint: put $u(t,x) = v(t,x)e^{\alpha t + \beta x}$.

6. Let $u(t,x,y)$ be activator concentration and $v(t,x,y)$ inhibitor concentration in a Gierer-Meinhardt activator-inhibitor model for animal coat pattern formation, which for two space dimensions is given by

$$\begin{cases} u_t = \dfrac{1}{4} - \dfrac{5u}{4} + \dfrac{u^2}{v} + D_1\,(u_{xx} + u_{yy}) \\ v_t = u^2 - v + D_2\,(v_{xx} + v_{yy}) \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability. Suppose now that $D_1 = 1/3$, $D_2 = 4$, that $0 < x < L_1 = 2\pi$, $0 < y < L_2 = 4\pi$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t}\cos\frac{m\pi x}{L_1}\cos\frac{n\pi y}{L_2}$. For what values of (m,n) can we have diffusive instabilities ($\sigma > 0$)? Sketch the resulting two-dimensional patterns.

TATM38, Math.Bio., 21/8 2023, solution sketches

(1) $\frac{dN}{dt} = f(N) = 2(1-\frac{N}{6})N - \frac{N^3}{3} = N(2-\frac{N}{3}-\frac{N^2}{3})$

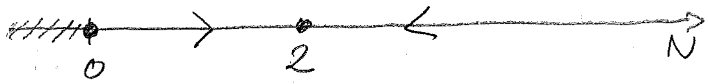
steady states $f(\bar N)=0 \Rightarrow \bar N_1 = 0$ or $\bar N^2+\bar N-6=0 \Rightarrow \bar N_2 = 2$ $(\bar N_3=-3<0)$

stable if $f'(\bar N_j)<0$. $f'(N) = 2-\frac{2N}{3}-N^2 \Rightarrow$

$f'(0) = 2>0 \Rightarrow \bar N_1 = 0$ unstable

$f'(2) = 2-\frac{4}{3}-4<0 \Rightarrow \bar N_2 = 2$ stable

Phase line



$N(t)\to \bar N_2 = 2$ as $t\to\infty$ if $N(0)>0$.

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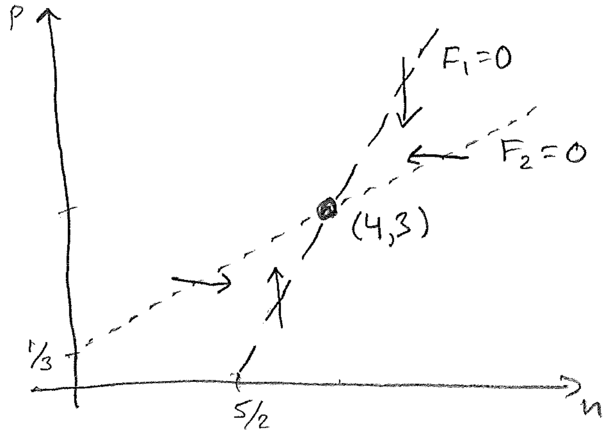
(2) Nullclines $F_1(n,p) = 5-2n+p = 0$ line

$F_2(n,p) = 1+2n-3p = 0$ line

$\begin{cases} F_1 = 0 \\ F_2 = 0 \end{cases} \Rightarrow (\bar n, \bar p) = (4,3)$ only steady state



Jacobian $J = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$

Eigenvalues $\lambda_1=-1, \lambda_2=-4 \Rightarrow$ (4,3) is stable

(or $\operatorname{Tr} J = -5<0$, $\det J = 4>0 \Rightarrow$ stable)

$\Rightarrow (n(t),p(t)) \to (4,3)$, $t\to\infty$

ODE for $n(t)$:

$n' = 5-2n+p \Rightarrow n'' = -2n'+p' = -2n'+1+2n-3p =$

$= -2n'+1+2n-3(n'-5+2n) = -5n'-4n+16 \Rightarrow$

$n''+5n'+4n = 16$ Char.eq. $r^2+5r+4=0 \Rightarrow r_1=-1, r_2=-4$

$\Rightarrow$ Homog. sol. is $n_h(t) = c_1e^{-t}+c_2e^{-4t}$

Try $n_p(t) = k$ (constant) for particular sol. $\Rightarrow n_p' = n_p'' = 0 \Rightarrow 0+0+4k = 16$

$\Rightarrow k = 4 \Rightarrow n_p(t) = 4$ and $n(t) = n_h(t)+n_p(t) = c_1e^{-t}+c_2e^{-4t}+4$

As $t\to\infty$, $n(t)\to 0+0+4 = 4$, which agrees with $\bar n = 4$ in the stable steady state $(\bar n,\bar p) = (4,3)$

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(3) $\begin{cases} x_{n+1} = x_n\left(2-\frac{x_n}{2}-\frac{y_n}{2}\right) \\ y_{n+1} = y_n\left(2-\frac{x_n}{4}-\frac{3y_n}{4}\right) \end{cases}$ Steadystates $\begin{cases} \bar{x} = x_{n+1} = x_n \\ \bar{y} = y_{n+1} = y_n \end{cases} \Rightarrow$

$\begin{cases} \bar{x} = \bar{x}\left(2-\frac{\bar{x}}{2}-\frac{\bar{y}}{2}\right) \\ \bar{y} = \bar{y}\left(2-\frac{\bar{x}}{4}-\frac{3\bar{y}}{4}\right) \end{cases} \Rightarrow \begin{cases} \bar{x}\left(1-\frac{\bar{x}}{2}-\frac{\bar{y}}{2}\right)=0 \\ \bar{y}\left(1-\frac{\bar{x}}{4}-\frac{3\bar{y}}{4}\right)=0 \end{cases}$ $\begin{matrix} (1) \Rightarrow \bar{x}=0 \text{ or } \bar{x}+\bar{y}=2 \\ (2) \end{matrix}$

$\left.\begin{matrix} \bar{x}=0 \text{ in } (2) \Rightarrow \bar{y}\left(1-\frac{3\bar{y}}{4}\right)=0 \Rightarrow \bar{y}=0 \text{ or } \bar{y}=4/3 \\ \bar{x}=2-\bar{y} \text{ in } (2) \Rightarrow \bar{y}\left(\frac{1}{2}-\frac{\bar{y}}{2}\right)=0 \Rightarrow \underset{\Downarrow}{\bar{y}=0} \text{ or } \underset{\Downarrow}{\bar{y}=1} \\ \qquad\qquad\qquad\qquad \bar{x}=2 \qquad \bar{x}=1 \end{matrix}\right\} \Rightarrow$

4 steady states $(0,0), (0,4/3), (2,0), (1,1)$, stable if $|\lambda_{1,2}|<1$ of J

$J(x,y) = \begin{pmatrix} 2-x-\frac{y}{2} & -\frac{x}{2} \\ -\frac{y}{4} & 2-\frac{x}{4}-\frac{3y}{2} \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1=\lambda_2=2 \Rightarrow$ unstable

$J(0,4/3) = \begin{pmatrix} 4/3 & 0 \\ -1/3 & 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 4/3 > 1 \\ \lambda_2 = 0 \end{cases} \Rightarrow$ unstable, $J(2,0) = \begin{pmatrix} 0 & -1 \\ 0 & 3/2 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 3/2 > 1 \\ \lambda_2 = 0 \end{cases} \Rightarrow$ unstable

$J(1,1) = \begin{pmatrix} 1/2 & -1/2 \\ -1/4 & 1/4 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 3/4 \\ \lambda_2 = 0 \end{cases} \Rightarrow$ stable (can also use $|\mathrm{Tr}\, J| < 1 + \det J < 2 \Rightarrow$ stable)

$\Rightarrow$ $(1,1)$ is stable, $(0,0), (0,4/3), (2,0)$ unstable

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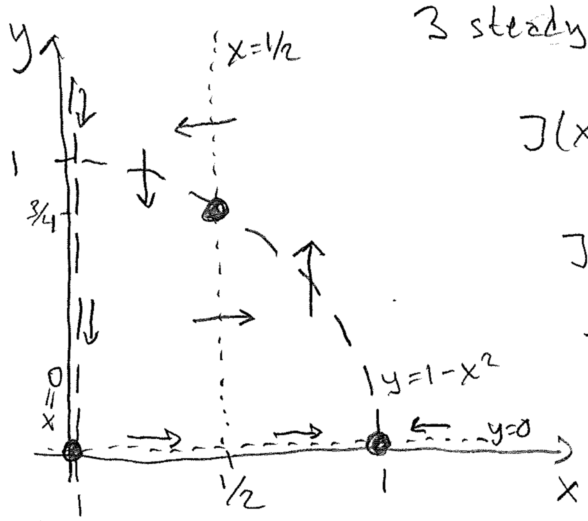
(4) $\begin{cases} x' = x(1-x^2-y) \\ y' = y(-1+2x) \end{cases}$ $\begin{matrix} x\text{-nullclines} & x=0 \text{ and } y=1-x^2 \\ y\text{-nullclines} & y=0 \text{ and } x=1/2 \end{matrix}$



3 steady states, $(\bar{x}_1,\bar{y}_1)=(0,0), (\bar{x}_2,\bar{y}_2)=(1,0), (\bar{x}_3,\bar{y}_3)=\left(\frac{1}{2},\frac{3}{4}\right)$

$J(x,y) = \begin{pmatrix} 1-3x^2-y & -x \\ 2y & -1+2x \end{pmatrix} \Rightarrow$

$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 1 > 0 \\ \lambda_2 = -1 \end{cases} \Rightarrow$ unstable (saddle)

$J(1,0) = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = 1 > 0 \\ \lambda_2 = -2 \end{cases} \Rightarrow$ unstable (saddle)

$J\left(\frac{1}{2},\frac{3}{4}\right) = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 0 \end{pmatrix} = J$ $\left.\begin{matrix} \mathrm{Tr}\, J = -\frac{1}{2} < 0 \\ \det J = \frac{3}{4} > 0 \end{matrix}\right\} \Rightarrow$ stable

[eigenvalues of J: $\lambda_{1,2} = \frac{-1 \pm i\sqrt{2}}{2}$ have $\mathrm{Re}(\lambda_{1,2}) = -\frac{1}{2} < 0 \Rightarrow$ stable (spiral)]

$(x(t),y(t)) \to \left(\frac{1}{2},\frac{3}{4}\right), t \to \infty$ if $x(0)>0$ and $y(0)>0$

$\big(x(0)=0 \Rightarrow (x(t),y(t)) \to (0,0)$, $y(0)=0 \Rightarrow (x(t),y(t)) \to (1,0)$ if $x(0)>0\big)$

(5) $u(t,x) = v(t,x)e^{\alpha t+\beta x} \iff v(t,x) = u(t,x)e^{-\alpha t-\beta x}$

$u_t = (v_t+\alpha v)e^{\alpha t+\beta x}$, $u_x = (v_x+\beta v)e^{\alpha t+\beta x}$, $u_{xx} = (v_{xx}+2\beta v_x+\beta^2 v)e^{\alpha t+\beta x} \Rightarrow$

$u_t = 3u_{xx}-6u_x \iff (v_t+\alpha v)e^{\alpha t+\beta x} = (3v_{xx}+6\beta v_x+3\beta^2 v-6v_x-6\beta v)e^{\alpha t+\beta x}$

$\iff v_t = 3v_{xx}+v_x(6\beta-6)+v(3\beta^2-6\beta-\alpha)$ Take $\beta=1$ and $\alpha=3\beta^2-6\beta=-3$

to get $v_t = 3v_{xx}$, $v(t,x) = u(t,x)e^{3t-x}$

IBVP for $v(t,x)$:

$$\begin{cases} v_t = 3v_{xx} \\ v(t,0) = u(t,0)e^{3t} = 0 \\ v(t,2\pi) = u(t,2\pi)e^{3t-2\pi} = 0 \\ v(0,x) = u(0,x)e^{-x} = 2\sin\frac{x}{2}-\sin\frac{3x}{2} \end{cases}$$

Separation of variables:

$v(t,x) = T(t)X(x) \Rightarrow$

$\frac{X''}{X} = \frac{T'}{3T} = \lambda = \text{constant}$

$\Rightarrow T(t) = e^{3\lambda t}$

$v(t,0) = v(t,2\pi) \Rightarrow X(0) = X(2\pi) = 0$

$$\left.\begin{matrix} X''(x)-\lambda X(x) = 0 \\ X(0) = X(2\pi) = 0 \end{matrix}\right\} \Rightarrow X_n(x) = \sin\frac{nx}{2},\ n=1,2,\dots,\ \lambda = -\frac{n^2}{4}$$

$\Rightarrow v(t,x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{nx}{2} e^{-3n^2t/4}$, $v(0,x) = \sum_{n=1}^{\infty}\alpha_n\sin\frac{nx}{2} = 2\sin\frac{x}{2}-\sin\frac{3x}{2}$

Identify $\alpha_1=2, \alpha_3=-1, \alpha_n=0\ n\neq 1,3 \Rightarrow v(t,x) = 2e^{-3t/4}\sin\frac{x}{2}-e^{-27t/4}\sin\frac{3x}{2}$

$\Rightarrow u(t,x) = v(t,x)e^{x-3t} = e^{x-3t}\left(2e^{-3t/4}\sin\frac{x}{2}-e^{-27t/4}\sin\frac{3x}{2}\right)$

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(6) $$\begin{cases} u_t = \frac{1}{4}-\frac{5u}{4}+\frac{u^2}{v}+D_1(u_{xx}+u_{yy}) \\ v_t = u^2-v+D_2(v_{xx}+v_{yy}) \end{cases}$$

Spatially uniform steady state $(\bar u,\bar v)$

$$\begin{cases} \frac{1}{4}-\frac{5\bar u}{4}+\frac{\bar u^2}{\bar v} = 0 \\ \bar u^2-\bar v = 0 \Rightarrow \frac{\bar u^2}{\bar v} = 1 \Rightarrow \bar u = 1 \Rightarrow \bar v = 1 \end{cases}$$

$(\bar u,\bar v) = (1,1)$

$J(u,v) = \begin{pmatrix} -5/4+\frac{2u}{v} & -\frac{u^2}{v^2} \\ 2u & -1 \end{pmatrix} \Rightarrow J(1,1) = \begin{pmatrix} 3/4 & -1 \\ 2 & -1 \end{pmatrix} = J$ $\left.\begin{matrix} \operatorname{Tr} J = -1/4 < 0 \\ \det J = 5/4 > 0 \end{matrix}\right\} \Rightarrow$ stable

Turing condition $J_{11}D_2+J_{22}D_1 > 2\sqrt{D_1D_2\det J} \iff \frac{3}{4}D_2-D_1 > \sqrt{5D_1D_2}$

$D_1 = \frac{1}{3}, D_2 = 4 \Rightarrow 3-\frac{1}{3} > \sqrt{\frac{20}{3}}$, satisfied $\left[\left(\frac{8}{3}\right)^2 = \frac{64}{9} > \frac{60}{9} = \frac{20}{3}\right]$

Unstable perturbations if

$$0 > \begin{vmatrix} 3/4-\frac{Q^2}{3} & -1 \\ 2 & -1-4Q^2 \end{vmatrix} = \frac{4}{3}Q^4-\frac{8}{3}Q^2+\frac{5}{4} = \frac{4}{3}\left(Q^4-2Q^2+\frac{15}{16}\right) = \frac{4}{3}\left((Q^2-1)^2-\frac{1}{16}\right) \iff$$

$(Q^2-1)^2 < \frac{1}{16} \iff -\frac{1}{4} < Q^2-1 < \frac{1}{4} \iff \frac{3}{4} < Q^2 < \frac{5}{4}$, where $Q^2 = \left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\pi^2 =$

$= \frac{m^2}{4}+\frac{n^2}{16} \Rightarrow \frac{3}{4} < \frac{m^2}{4}+\frac{n^2}{16} < \frac{5}{4} \iff 12 < 4m^2+n^2 < 20$, true for $(m,n) = (0,4),(1,3),(2,0),$ $(2,1)$

Patterns