Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2023-10-28 , 8.00 - 13.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. The interaction between two species $x(t)$ and $y(t)$ is modeled by the linear system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

Determine the general solution to this system and draw a picture of the entire phase plane (including negative x and y). Is the steady state (= equilibrium point) $(0,0)$ stable?
What is the solution if the initial values are $x(0) = 1$ and $y(0) = 4$?
Show that $y(t) \approx x(t)$ for large t for any initial levels $x(0) > 0$, $y(0) > 0$.

2. Consider the logistic map $\quad x_{n+1} = r\, x_n(1 - x_n)\ ,\ n = 0, 1, 2, \ldots$ with $0 \le x_0 \le 1$.
a) Let $r = 2.5$. Find the steady states (= equilibrium points) and determine their stability. Sketch a cobweb diagram for $0 \le x \le 1$ for some x_0 with $0 < x_0 < 1$. What happens to x_n as $n \to \infty$ if $0 < x_0 < 1$?
b) Let $r = 1 + \sqrt{5}$ (≈ 3.23). Find the steady states and determine their stability. Show that $\tilde{x} = \frac{1}{2}$ is one of the points of a period-2 oscillation between $\tilde{x}$ and some $\hat{x}$. What is $\hat{x}$?

3. Two populations $N(t)$ and $P(t)$ compete for the same resources, and their time evolution is given by

$$\begin{cases} \dfrac{dN}{dt} = N(2 - N - P) \\ \dfrac{dP}{dt} = P(3 - 3P - N) \end{cases}$$

Find all steady states, determine their stability, and draw a phase plane picture with nullclines and directions of the vector field. What happens to the populations as $t \to \infty$ if $N(0) > 0$ and $P(0) > 0$? Is coexistence possible?

PLEASE TURN

4. Let x_n and y_n be the numbers of prey and predators, respectively, at time n in a time discrete Lotka-Volterra type predator-prey model

$$\begin{cases} x_{n+1} = x_n + x_n(a - by_n) \\ y_{n+1} = y_n + y_n(-c + dx_n) \end{cases}$$

where $a > 0, b > 0, c > 0, d > 0$ are constants. Show that all steady states are unstable.

5. For $0 < x < \pi$ and $t > 0$, solve the initial-boundary value problem (IBVP) for $u(t, x)$

$$\begin{cases} u_t = u_{xx} + 3u \\ u_x(t, 0) = 0 \\ u_x(t, \pi) = 0 \\ u(0, x) = 2\cos 2x - \cos 3x \end{cases}$$

Hint: put $u(t, x) = v(t, x)e^{\alpha t}$.

6. A model with spatial diffusion for two chemical concentrations $u(t, x, y)$ and $v(t, x, y)$ in two space dimensions is given by

$$\begin{cases} u_t = \dfrac{3}{2} - uv + D_1(u_{xx} + u_{yy}) \\ v_t = uv - v - \dfrac{1}{2} + D_2(v_{xx} + v_{yy}) \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability. Suppose now that $D_1 = 2$, $D_2 = \frac{1}{10}$, that $0 < x < L_1 = 2\pi$, $0 < y < L_2 = 2\sqrt{2}\,\pi$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t} \cos \frac{m\pi x}{L_1} \cos \frac{n\pi y}{L_2}$. For what values of (m, n) can we have diffusive instabilities ($\sigma > 0$)? Sketch the resulting two-dimensional patterns.

TATM38, 28/10 2023, solution sketches

(1) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ Eigenvalues $\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 9 = 0 \Rightarrow \begin{cases} \lambda_1 = 3 > 0 \\ \lambda_2 = -3 < 0 \end{cases}$

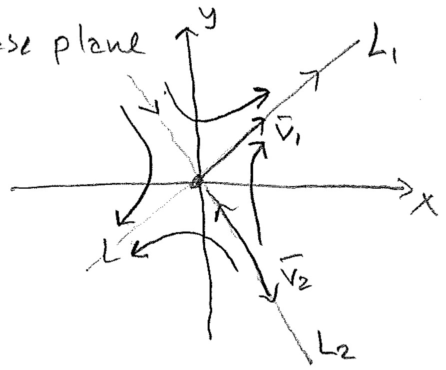
$\Rightarrow (0,0)$ is a saddle point (unstable)

Eigenvectors $\lambda_1 = 3 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -3 \Rightarrow \bar{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow$ general solution is

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -1 \end{cases} \Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 3e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Phase plane



$e^{-3t} \to 0,\ t \to \infty \Rightarrow$ For large t,

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x(t) \approx y(t)$

(unless $c_1 = 0$ which happens at L_2 but not if $x(0) > 0,\ y(0) > 0$)

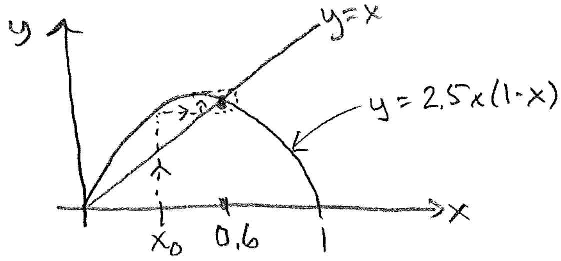
(2) a) $x_{n+1} = f(x_n) = 2.5x_n(1 - x_n)$

Steady states $\bar{x} = 2.5\bar{x}(1-\bar{x}) \Rightarrow \bar{x}_1 = 0,\ \bar{x}_2 = 0.6$

Stability $f'(x) = 2.5 - 5x \Rightarrow |f'(0)| = 2.5 > 1 \Rightarrow \bar{x}_1 = 0$ unstable

$|f'(0.6)| = |-0.5| = 0.5 < 1 \Rightarrow \bar{x}_2 = 0.6$ stable

Cobweb



For $0 < x_0 < 1$, $x_n \to 0.6$ as $n \to \infty$

b) $x_{n+1} = f(x_n) = (1+\sqrt{5})x_n(1-x_n)$

Steady states $\bar{x} = (1+\sqrt{5})\bar{x}(1-\bar{x}) \Rightarrow \bar{x}_1 = 0,\ \bar{x}_2 = 1 - \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}}{1+\sqrt{5}}$

$f'(x) = (1+\sqrt{5})(1-2x)$ $\quad |f'(0)| = 1+\sqrt{5} > 1 \Rightarrow \bar{x}_1 = 0$ unstable

$\left|f'\left(\frac{\sqrt{5}}{1+\sqrt{5}}\right)\right| = \left|(1+\sqrt{5})\left(1 - \frac{2\sqrt{5}}{1+\sqrt{5}}\right)\right| = |1+\sqrt{5}-2\sqrt{5}| = |1-\sqrt{5}| =$

$\tilde{x} = \frac{1}{2} \Rightarrow$ $\quad = \sqrt{5} - 1 > 1 \Rightarrow \bar{x}_2 = \frac{\sqrt{5}}{1+\sqrt{5}}$ unstable

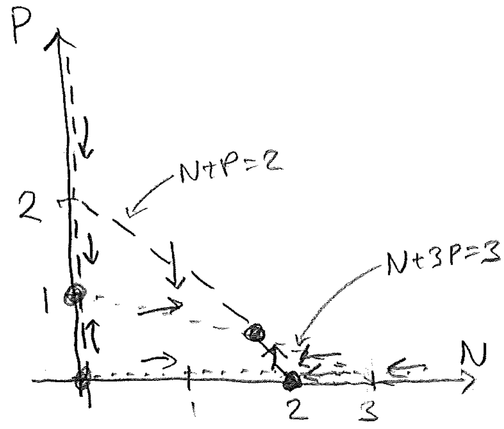
$f(\tilde{x}) = (1+\sqrt{5}) \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1+\sqrt{5}}{4} = \hat{x}$

$f(\hat{x}) = (1+\sqrt{5})\left(\frac{1+\sqrt{5}}{4}\right)\left(1 - \frac{1+\sqrt{5}}{4}\right) = \frac{1}{4}(6+2\sqrt{5})\left(\frac{3-\sqrt{5}}{4}\right) = \frac{1}{8}(3+\sqrt{5})(3-\sqrt{5}) = \frac{1}{2} = \tilde{x}$

So $f(\tilde{x}) = \hat{x}$ and $f(\hat{x}) = \tilde{x} \Rightarrow$ period-2 oscillation between $\tilde{x}$ and $\hat{x}$.

$\hat{x} = \frac{1+\sqrt{5}}{4}$

(3) $\begin{cases} N' = N(2-N-P) \\ P' = P(3-3P-N) \end{cases}$ N nullclines $N=0$, $N+P=2$ (dashed)
P nullclines $P=0$, $N+3P=3$ (dotted)



4 steady states $(\bar N_1,\bar P_1)=(0,0)$
$(\bar N_2,\bar P_2)=(2,0)$
$(\bar N_3,\bar P_3)=(0,1)$

$\left.\begin{matrix} N+P=2 \\ N+3P=3 \end{matrix}\right\} \xrightarrow{\text{gives}} (\bar N_4,\bar P_4)=\left(\frac{3}{2},\frac{1}{2}\right)$

$$J(N,P)=\begin{pmatrix} 2-2N-P & -N \\ -P & 3-6P-N \end{pmatrix}$$

$J(0,0)=\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $\left.\begin{matrix} \lambda_1=2>0 \\ \lambda_2=3>0 \end{matrix}\right\} \Rightarrow$ unstable

$J(2,0)=\begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$ $\left.\begin{matrix} \lambda_1=-2<0 \\ \lambda_2=1>0 \end{matrix}\right\} \Rightarrow$ unstable (saddle), $J(0,1)=\begin{pmatrix} 1 & 0 \\ -1 & -3 \end{pmatrix}$ $\left.\begin{matrix} \lambda_1=1>0 \\ \lambda_2=-3<0 \end{matrix}\right\} \Rightarrow$ unstable (saddle)

$J\left(\frac{3}{2},\frac{1}{2}\right)=\begin{pmatrix} -3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix} \Rightarrow \lambda_{1,2}=\frac{-3\pm\sqrt{3}}{2}<0 \Rightarrow$ stable [or $\left.\begin{matrix} \operatorname{Tr} J=-3<0 \\ \det J=6/4>0 \end{matrix}\right\} \Rightarrow$ stable]

As $t\to\infty$, if $N(0)>0$ and $P(0)>0$, $\underline{(N,P)\to\left(\frac{3}{2},\frac{1}{2}\right)}$, which means that both populations will coexist.

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(4) $\begin{cases} x_{n+1} = x_n + x_n(a-by_n) \\ y_{n+1} = y_n + y_n(-c+dx_n) \end{cases}$ Steady states $\begin{cases} \bar x = \bar x + \bar x(a-b\bar y) \\ \bar y = \bar y + \bar y(-c+d\bar x) \end{cases}$

$\Leftrightarrow \begin{cases} \bar x(a-b\bar y)=0 & (1) \\ \bar y(-c+d\bar x)=0 & (2) \end{cases}$ $(1)\Rightarrow \bar x=0 \overset{(2)}{\Downarrow} \bar y=0$ or $\bar y=a/b \overset{(2)}{\Downarrow} \bar x=c/d$ $\Rightarrow$ Two steady states $(0,0)$ and $\left(\frac{c}{d},\frac{a}{b}\right)$

$J(x,y)=\begin{pmatrix} 1+a-by & -bx \\ dy & 1-c+dx \end{pmatrix} \Rightarrow$ Stable if J has $|\lambda_{1,2}|<1$

$J(0,0)=\begin{pmatrix} 1+a & 0 \\ 0 & 1-c \end{pmatrix}$ with eigenvalues $\begin{cases} \lambda_1=1+a \\ \lambda_2=1-c \end{cases}$ $|\lambda_1|=\lambda_1=1+a>1$ $\Rightarrow (0,0)$ unstable

$J\left(\frac{c}{d},\frac{a}{b}\right)=\begin{pmatrix} 1 & -\frac{bc}{d} \\ \frac{ad}{b} & 1 \end{pmatrix}$ eigenvalues $(\lambda-1)^2+\frac{ad}{b}\cdot\frac{bc}{d}=(\lambda-1)^2+ac=0$

$\Rightarrow \lambda_{1,2}=1\pm i\sqrt{ac}$, $|\lambda_{1,2}|=\sqrt{1+ac}>1$
$\Rightarrow \left(\frac{c}{d},\frac{a}{b}\right)$ unstable

[or check if $\underbrace{|\operatorname{Tr} J|}_{2}<\underbrace{1+\underbrace{\det J}_{1+ac}}_{2+ac}<2$; Jury test — not satisfied]

So, there are two steady states and both are unstable.

(5) $u(t,x)=v(t,x)e^{\alpha t} \Rightarrow u_t=(v_t+\alpha v)e^{\alpha t},\ u_x=v_x e^{\alpha t},\ u_{xx}=v_{xx}e^{\alpha t}$

$\Rightarrow u_t=u_{xx}+3u \Leftrightarrow (v_t+\alpha v)e^{\alpha t}=(v_{xx}+3v)e^{\alpha t}$. With $\alpha=3$ we get

$v_t=v_{xx}$ and the IBVP for $v(t,x)$:

$$\begin{cases} v_t=v_{xx}, & 0<x<\pi,\ t>0 \\ v_x(t,0)=u_x(t,0)e^{-3t}=0 \\ v_x(t,\pi)=u_x(t,\pi)e^{-3t}=0 \\ v(0,x)=u(0,x)e^{-3\cdot 0}=u(0,x)=2\cos 2x-\cos 3x \end{cases}$$

Separation of variables $v(t,x)=T(t)X(x)$

$\Rightarrow \frac{T'(t)}{T(t)}=\frac{X''(x)}{X(x)}=\lambda=\text{constant}$

$\Rightarrow T(t)=e^{\lambda t}$

$$\left.\begin{array}{l} v_x(t,0)=T(t)X'(0)=0 \\ v_x(t,\pi)=T(t)X'(\pi)=0 \end{array}\right\} \Rightarrow X'(0)=X'(\pi)=0$$

$$\left.\begin{array}{l} X''(x)-\lambda X(x)=0 \\ X'(0)=X'(\pi)=0 \end{array}\right\} \Rightarrow X_n(x)=\cos nx,\ n=0,1,2,\dots \text{ (non-zero solutions)}$$

$\lambda=-n^2$ so $T_n(t)=e^{-n^2 t}$

Linear PDE and homogeneous boundary conditions $\Rightarrow$

$v(t,x)=\sum_{n=0}^{\infty}\alpha_n e^{-n^2 t}\cos nx$ solves $\begin{cases} v_t=v_{xx} \\ v_x(t,0)=v_x(t,\pi)=0 \end{cases}$ $\forall \{\alpha_n\}_0^\infty$

Initial condition $v(0,x)=\sum_{n=0}^{\infty}\alpha_n\cos nx=2\cos 2x-\cos 3x \Rightarrow \alpha_2=2, \alpha_3=-1$ and $\alpha_n=0$ for $n\neq 2,3$

$\Rightarrow v(t,x)=2e^{-4t}\cos 2x-e^{-9t}\cos 3x$

and $u(t,x)=e^{3t}v(t,x)=\underline{2e^{-t}\cos 2x-e^{-6t}\cos 3x}$

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(6) $\begin{cases} u_t=\frac{3}{2}-uv+D_1(u_{xx}+u_{yy}) \\ v_t=uv-v-\frac{1}{2}+D_2(v_{xx}+v_{yy}) \end{cases}$

Spatially uniform steady state $(\bar u,\bar v)$: $\left.\begin{array}{l}\frac{3}{2}-\bar u\bar v=0 \\ \bar u\bar v-\bar v-\frac12=0\end{array}\right\} \Rightarrow (\bar u,\bar v)=(\frac32,1)$

Stability for $D_1=D_2=0$

$J(u,v)=\begin{pmatrix}-v & -u \\ v & u-1\end{pmatrix} \Rightarrow J(\frac32,1)=\begin{pmatrix}-1 & -3/2 \\ 1 & 1/2\end{pmatrix}$ $\left.\begin{array}{l}\mathrm{Tr}\, J=-\frac12<0 \\ \det J=1>0\end{array}\right\} \Rightarrow \underline{\text{stable}}$

Turing condition $J_{11}D_2+J_{22}D_1>2\sqrt{D_1D_2\det J} \Leftrightarrow \underline{-D_2+\frac12 D_1>2\sqrt{D_1D_2}}$

$D_1=2, D_2=\frac{1}{10}$: $1-\frac{1}{10}=\frac{9}{10}>2\sqrt{\frac{2}{10}}=\frac{2}{\sqrt5}$ (since $(\frac{9}{10})^2=\frac{81}{100}>\frac{80}{100}=\frac45=(\frac{2}{\sqrt5})^2$) Turing condition satisfied

Unstable perturbations if

$0>\det(J-Q^2D)=\begin{vmatrix}-1-2Q^2 & -3/2 \\ 1 & \frac12-\frac{1}{10}Q^2\end{vmatrix}=\frac15 Q^4-\frac{9}{10}Q^2+1=\frac15\left(Q^4-\frac92 Q^2+5\right)=\frac15\left((Q^2-\frac94)^2-\frac{1}{16}\right)$

$\Leftrightarrow |Q^2-\frac94|<\frac14 \Leftrightarrow -\frac14<Q^2-\frac94<\frac14 \Leftrightarrow 2<Q^2<\frac52$

$Q^2=\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\pi^2=\frac{m^2}{4}+\frac{n^2}{8} \Rightarrow 2<\frac{m^2}{4}+\frac{n^2}{8}<\frac52 \Leftrightarrow \underline{16<2m^2+n^2<20}$

Possible integer solutions $(m,n)=(1,4),(2,3),(3,0),(3,1)$. Patterns: