Linköpings universitet
Matematiska institutionen

Kurskod: TATM38
Provkod: TEN1

Written examination, TATM38 Mathematical Models in Biology

2024-01-04 , 14.00 - 19.00

Each problem is worth 4 points. To obtain a grade 3, 4 or 5, you need 10, 14 or 18 points, respectively. You must not use any aids (no textbooks, notes, calculators or other electronic tools).

1. A population $N(t)$ is described by a modified logistic equation

$$\frac{dN}{dt} = rN\left(2 - \sqrt{\frac{N}{B}} - \frac{N}{B}\right)$$

Here $r > 0$ and $B > 0$ are constants.
Find all steady states (= equilibrium points) and determine their stability. Sketch the phase line for $N \geq 0$. What happens to the population as $t \to \infty$?

2. In a genetic model, let x_n and y_n be the numbers of alleles A and B in generation n. Suppose that 20% of A mutate into B in one time step, and 10% of B mutate into A. This gives the linear equations

$$\begin{cases} x_{n+1} = 0.8x_n + 0.1y_n \\ y_{n+1} = 0.2x_n + 0.9y_n \end{cases}$$

for $n = 0, 1, 2, \ldots$. Find the general solution of this system. What is the solution if $x_0 = 100$ and $y_0 = 50$? Find the limits of x_n and y_n as $n \to \infty$. What percentages of A and B will there be in the system as $n \to \infty$?

3. A simplified chemostat model, with bacteria and nutrient concentrations $N(t)$ and $C(t)$, respectively, is decribed by

$$\begin{cases} \dfrac{dN}{dt} = 2CN - N \\[2mm] \dfrac{dC}{dt} = -CN - C + 2 \end{cases}$$

Find all steady states, determine their stability, and draw a phase plane picture for $N \geq 0$ and $C \geq 0$ with nullclines and directions of the vector field. What are the limits of $N(t)$ and $C(t)$ as $t \to \infty$ if $N(0) > 0$?

PLEASE TURN

4. Let $S(t)$, $I(t)$ and $R(t)$ be the numbers of susceptibles, infective, and removed, respectively, in a SIRS epidemic model. The total population N is constant, $N = 1000$, and with $R(t) = N - S(t) - I(t)$ it is sufficient to study a two-dimensional system for $S(t)$ and $I(t)$. With certain choices of parameter values, the system is

$$\begin{cases} \dfrac{dS}{dt} &= \dfrac{1000 - S - I}{4} - \dfrac{SI}{200} \\ \dfrac{dI}{dt} &= \dfrac{SI}{200} - \dfrac{I}{2} \end{cases}$$

Find all steady states of the system and determine their stability. Draw a phase plane picture (with nullclines and directions of the vector field). Note that $S \geq 0$, $I \geq 0$, and $S + I \leq 1000$. What happens to the number of susceptibles and infective as $t \to \infty$ if $I(0) > 0$?

5. For $0 < x < 1$, $0 < y < 2$, and $t > 0$, solve the initial-boundary value problem (IBVP) in two space-dimensions for $u(t, x, y)$

$$\begin{cases} 5u_t = u_{xx} + u_{yy} \\ u(t,0,y) = u(t,1,y) = 0 \\ u(t,x,0) = u(t,x,2) = 0 \\ u(0,x,y) = \sin(2\pi x)\sin(\frac{3\pi y}{2}) + 3\sin(3\pi x)\sin(\pi y) \end{cases}$$

6. Consider a two-component reaction-diffusion system in one space dimensions with concentrations $u(t,x)$ and $v(t,x)$:

$$\begin{cases} u_t = \frac{3}{2}u^2 v - u + \frac{1}{2} + D_1\, u_{xx} \\ v_t = \frac{1}{4}(1 - u^2 v) + D_2\, v_{xx} \end{cases}$$

Find the spatially uniform steady state $(\bar{u}, \bar{v})$, and show that it is stable if there is no diffusion ($D_1 = D_2 = 0$).
With diffusion present ($D_1 > 0, D_2 > 0$), find the condition for Turing diffusive instability. Suppose now that $D_1 = 1$, $D_2 = 20$, that $0 < x < L = 15\pi$, and that perturbations near $(\bar{u}, \bar{v})$ have the form $e^{\sigma t} \cos \frac{n\pi x}{L}$. For what values of n can we have diffusive instabilities ($\sigma > 0$)? Sketch the resulting one-dimensional patterns.

TATM38, 4/1 2024, solution sketches

(1) $N' = f(N) = rN\left(2-\sqrt{\frac{N}{B}}-\frac{N}{B}\right)$, steady states $f(\bar{N})=0$: $\bar{N}_1 = 0$ or

$2-\sqrt{\frac{N}{B}}-\frac{N}{B}=0$, $x=\sqrt{\frac{N}{B}} \Rightarrow x^2+x-2 \Rightarrow x=1$ $(x=-2<0)$ $\Rightarrow \bar{N}_2 = B$

Stable if $f'(\bar{N}_j)<0$ $\quad f'(N) = r\left(2-\frac{3}{2}\sqrt{\frac{N}{B}}-2\frac{N}{B}\right) \Rightarrow$

$f'(0) = 2r > 0 \Rightarrow \bar{N}_1 = 0$ unstable

$f'(B) = r(2-\frac{3}{2}-2) = -\frac{3r}{2} < 0 \Rightarrow \bar{N}_2 = B$ stable, $N(t)\to B$ as $t\to\infty$ if $N(0)>0$

$(N(0)=0 \Rightarrow N(t)=0$ for all $t)$

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(2) $\begin{pmatrix} x_{n+1} \\ y_{n+1}\end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}}_{A}\begin{pmatrix} x_n \\ y_n\end{pmatrix}$. Eigenvalues $\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = \lambda^2-1.7\lambda+0.7=0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 0.7$. Eigenvectors $\lambda_1 = 1 \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 2\end{pmatrix}$, $\lambda_2 = 0.7 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1\end{pmatrix}$

General solution

$\begin{pmatrix} x_n \\ y_n\end{pmatrix} = c_1\lambda_1^n\bar{v}_1 + c_2\lambda_2^n\bar{v}_2 = c_1\begin{pmatrix} 1 \\ 2\end{pmatrix} + c_2\underbrace{(0.7)^n}_{\to 0,\, n\to\infty}\begin{pmatrix} 1 \\ -1\end{pmatrix} \to c_1\begin{pmatrix} 1 \\ 2\end{pmatrix}$, $n\to\infty$

$\left.\begin{matrix} x_0 = 100 \\ y_0 = 50\end{matrix}\right\} \overset{(n=0)}{\Rightarrow} \left.\begin{matrix} c_1+c_2 = 100 \\ 2c_1 - c_2 = 50\end{matrix}\right\} \Rightarrow c_1 = c_2 = 50 \Rightarrow \begin{pmatrix} x_n \\ y_n\end{pmatrix} = 50\begin{pmatrix} 1 \\ 2\end{pmatrix} + 50\cdot(0.7)^n\begin{pmatrix} 1 \\ -1\end{pmatrix}$

For $n\to\infty$, $\begin{pmatrix} x_n \\ y_n\end{pmatrix} \approx c_1\begin{pmatrix} 1 \\ 2\end{pmatrix} = 50\begin{pmatrix} 1 \\ 2\end{pmatrix} = 150\begin{pmatrix} 1/3 \\ 2/3\end{pmatrix}$, so $\frac{1}{3}\approx 33\%$ have allele A

and $\frac{2}{3}\approx 67\%$ have allele B

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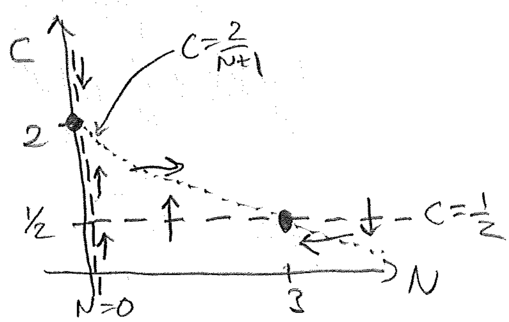
(3) $\begin{cases} N' = 2CN - N & = N(2C-1) \\ C' = -CN - C + 2 & \end{cases}$ N nullclines $N=0$, $C=1/2$ (2 lines, dashed)

C nullclines $C(N+1)=2$, $C = \frac{2}{N+1}$ (dotted)

Steady states $N=0 \Rightarrow C=2$, $C=\frac{1}{2} \Rightarrow N=3 \Rightarrow$ 2 steady states:

$(\bar{N}_1, \bar{C}_1) = (0,2)$, $(\bar{N}_2,\bar{C}_2) = (3, 1/2)$



$J(N,C) = \begin{pmatrix} 2C-1 & 2N \\ -C & -N-1\end{pmatrix} \Rightarrow$

$J(0,2) = \begin{pmatrix} 3 & 0 \\ -2 & -1\end{pmatrix}$ $\left.\begin{matrix}\lambda_1 = 3>0 \\ \lambda_2 = -1 < 0\end{matrix}\right\}$ saddle point (unstable)

$J(3,\frac{1}{2}) = \begin{pmatrix} 0 & 6 \\ -1/2 & -4\end{pmatrix} = J_2$ $\left.\begin{matrix}\operatorname{Tr} J_2 = -4 < 0 \\ \det J_2 = 3 > 0\end{matrix}\right\} \Rightarrow$ stable

(or $\lambda_1 = -1 < 0$, $\lambda_2 = -3 < 0 \Rightarrow$ stable)

When $t\to\infty$, $(N(t), C(t)) \to (3, \frac{1}{2})$ if $N(0)>0$ $\left(N(0)=0 \Rightarrow (N(t),C(t))\to(0,2)\right)$

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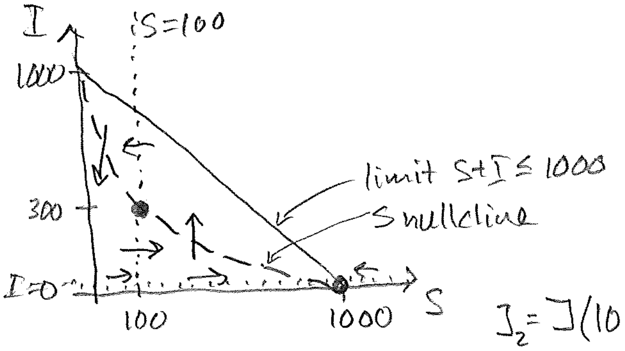
(4) $\begin{cases} S' = \dfrac{1000-S-I}{4} - \dfrac{SI}{200} \\ I' = \dfrac{SI}{200} - \dfrac{I}{2} = \dfrac{I}{2}\left(\dfrac{S}{100}-1\right) \end{cases}$ nullcline $\frac{I}{4}\left(1+\frac{S}{50}\right) = \frac{1000-S}{4} \Rightarrow I = \frac{1000-S}{1+S/50} = 50\cdot\frac{1000-S}{S+50}$ (dashed)

nullclines $I=0, S=100$ (dotted)

Steady states $I=0 \Rightarrow S=1000$, $S=100 \Rightarrow I=300 \Rightarrow (\bar S_1, \bar I_1) = (1000,0), (\bar S_2, \bar I_2) = (100,300)$

S nullcline $I = 50\,\frac{1000-S}{S+50}$ is a curve through $(0,1000)$, $(1000,0)$ and $(100,300)$



$$J(S,I) = \begin{pmatrix} -\frac14 - \frac{I}{200} & -\frac14 - \frac{S}{200} \\ \frac{I}{200} & \frac{S}{200} - \frac12 \end{pmatrix}$$

$J(1000,0) = \begin{pmatrix} -1/4 & -21/4 \\ 0 & 9/2 \end{pmatrix}$ $\left.\begin{matrix} \lambda_1 = \frac92 > 0 \\ \lambda_2 = -\frac14 < 0 \end{matrix}\right\}$ unstable, saddle point

$J_2 = J(100,300) = \begin{pmatrix} -7/4 & -3/4 \\ 3/2 & 0 \end{pmatrix}$ $\left.\begin{matrix} \mathrm{Tr}\,J_2 = -\frac74 < 0 \\ \det J_2 = \frac98 > 0 \end{matrix}\right\} \Rightarrow$ stable $\left(\lambda_{1,2} = \frac{-7 \pm i\sqrt{23}}{8},\ \mathrm{Re}(\lambda_{1,2}) < 0 \Rightarrow \text{stable}\right)$

If $I(0) > 0$, $(S(t), I(t)) \to (100,300)$ as $t\to\infty$

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(5) $\begin{cases} 5u_t = u_{xx} + u_{yy} \\ u(t,0,y) = u(t,1,y) = 0 \\ u(t,x,0) = u(t,x,2) = 0 \\ u(0,x,y) = \sin(2\pi x)\sin\left(\frac{3\pi y}{2}\right) + 3\sin(3\pi x)\sin(\pi y) \end{cases}$ (BC) (IC)

Separation of variables $u(t,x,y) = T(t)X(x)Y(y)$

$\Rightarrow \frac{5T'}{T} = \underbrace{\frac{X''}{X}}_{=\lambda} + \underbrace{\frac{Y''}{Y}}_{=\mu}$ (BC) $\Rightarrow$ $X(0)=X(1)=0$, $Y(0)=Y(2)=0$

$\left.\begin{matrix} X'' - \lambda X = 0 \\ X(0) = X(1) = 0 \end{matrix}\right\} \Rightarrow X_n(x) = \sin(n\pi x)$, $n = 1,2,\dots$, $\lambda = -n^2\pi^2$ $\quad \left.\begin{matrix} Y'' - \mu Y = 0 \\ Y(0) = Y(2) = 0 \end{matrix}\right\} \Rightarrow Y_m(y) = \sin\left(\frac{m\pi y}{2}\right)$, $m = 1,2,\dots$, $\mu = -m^2\pi^2/4$

$\Rightarrow T' = \frac{\lambda+\mu}{5} T = -\frac{\pi^2}{5}\left(n^2 + \frac{m^2}{4}\right)T \Rightarrow T_{n,m}(t) = e^{-\pi^2(4n^2+m^2)t/20} \Rightarrow$

$u(t,x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{n,m} e^{-\pi^2(4n^2+m^2)t/20} \sin(n\pi x)\sin\left(\frac{m\pi y}{2}\right) \underset{(IC)}{\Longrightarrow}$

$u(0,x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{n,m}\sin(n\pi x)\sin\left(\frac{m\pi y}{2}\right) = \sin(2\pi x)\sin\left(\frac{3\pi y}{2}\right) + 3\sin(3\pi x)\sin(\pi y) \Rightarrow$

$\alpha_{2,3} = 1$, $\alpha_{3,2} = 3$, all other $\alpha_{n,m} = 0 \Rightarrow u(t,x,y) = e^{-5\pi^2 t/4}\sin(2\pi x)\sin\left(\frac{3\pi y}{2}\right) + 3e^{-2\pi^2 t}\sin(3\pi x)\sin(\pi y)$

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(6) Spatially uniform steady states : $\begin{cases} \frac32 \bar u^2 \bar v - \bar u + \frac12 = 0 \\ \frac14(1 - \bar u^2 \bar v) = 0 \Rightarrow \bar u^2 \bar v = 1 \end{cases}$ $\Rightarrow \frac32 - \bar u + \frac12 = 0 \Rightarrow \bar u = 2 \Rightarrow \bar v = \frac14 \Rightarrow (\bar u, \bar v) = (2, 1/4)$

$J(u,v) = \begin{pmatrix} 3uv - 1 & \frac32 u^2 \\ -\frac12 uv & -\frac{u^2}{4} \end{pmatrix} \Rightarrow A = J(\bar u, \bar v) = J(2, 1/4) = \begin{pmatrix} 1/2 & 6 \\ -1/4 & -1 \end{pmatrix}$

$\left.\begin{matrix} \mathrm{Tr}\,A = -\frac12 < 0 \\ \det A = 1 > 0 \end{matrix}\right\} \Rightarrow$ stable if no diffusion

Turing condition $a_{11}D_2 + a_{22}D_1 > 2\sqrt{D_1 D_2 \det A} \iff \frac12 D_2 - D_1 > 2\sqrt{D_1 D_2}$

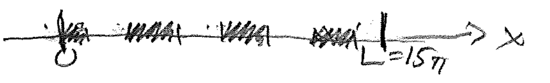
With $D_1 = 1, D_2 = 20$, $\frac12 D_2 - D_1 = 9 > 2\sqrt{20} = 2\sqrt{D_1 D_2}$ (since $9^2 = 81 > 80 = (2\sqrt{20})^2$)

Then, with $u = \frac{a_{11}D_2 + a_{22}D_1}{D_1 D_2} = \frac{9}{20}$, $w = \frac{\det A}{D_1 D_2} = \frac{1}{20}$ and $\Delta = \sqrt{\left(\frac{u}{2}\right)^2 - w} = \sqrt{\frac{81-80}{1600}} = \frac{1}{40}$,

unstable perturbations appear for $\frac u2 - \Delta < q^2 < \frac u2 + \Delta \iff \frac{8}{40} < q^2 < \frac{10}{40} \iff$

$\frac15 < q^2 < \frac14$, where $q^2 = \frac{n^2\pi^2}{L^2} = \frac{n^2}{15^2} = \frac{n^2}{225} \Rightarrow \frac{225}{5} < n^2 < \frac{225}{4} \iff 45 < n^2 < 56.25$

$n = 7$ only solution, pattern



▨ : $\cos\left(\frac{7\pi x}{15\pi}\right) > 0$