WASP course "Nonlinear Optimization"
Assignment-1

You will have several assignments, this is the first one. To pass the exam, you have to get 1/3 of points in total or to solve in every assignment at least 6 problems.

Rules. You can use any materials you want, just not a direct help from someone on how to solve a particular problem. You should send your solutions either electronically in a pdf format or in a paper form. If it is hand written, make sure it is readable.

The deadline to send solutions is February 16, 2022. Don't hesitate to ask me by email if something is not clear.

1. **(2 point)** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) = \sum_{i=1}^m \log(1 + e^{\mathbf{a}_i^\top \mathbf{x}})$, where $\mathbf{a}_i$ are given vectors. Compute $\nabla f(\mathbf{x})$.
2. **(2 points)** Compute the gradient and Hessian of $f(\mathbf{x}) = \log \sum_{i=1}^n e^{x_i}$.
3. **(2 point)** Compute the gradient ∇f of $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\mathbf{x}^\top - A\|^2$, where $A \in \mathbb{R}^{n\times n}$ is given.
4. **(2 points)** Prove that for a differentiable function $f\colon \mathbb{R}^d \to \mathbb{R}$ inequality
$$f(\mathbf{y}) \geqslant f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x}\rangle \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$$
implies convexity of f. (At the lecture we proved the opposite direction, so now we know that the two definitions are equivalent for differentiable functions).
5. **(2 points)** Prove Bernoulli inequality $(1+x)^r \geqslant 1 + rx$ for $r > 1$ and $x \geqslant -1$ in "one line".
6. **(3 points)** Another way to denote a partial derivative $\frac{\partial f(x,y)}{\partial x}$ is f_x. Similarly $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ is denoted by f_{xy}, etc. Given a function $f(x,y) = \sin x + x^6 y^{10} \cos y$, compute $f_{xxxxxyxxxxx}$.
7. **(2 points)** Let $f(x,y) = \sin(\pi x y^2)$. What is the linear approximation of f at the point $(1,1)$? Compare the values of f and its linear approximation at this point.
8. **(2 points)** Consider f from the previous example. What is the quadratic approximation of f at this point? Compare the values of f and its quadratic approximation at this point.
9. **(2 points)** Let $f(x,y) = \sin(x^2 + y) + y$. Find all critical points of f.
10. **(3 points)** Solve the following problem
$$\min_{x_1>0, x_2>0} \frac{1}{x_1 x_2} + x_1 + x_2.$$
11. **(3 points)** Consider the quadratic function $f(x) = \frac{1}{2}\langle A\mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{b}, \mathbf{x}\rangle + c$, where A is a symmetric $n \times n$ matrix. Prove that if the function f is bounded from below on $\mathbb{R}^n$, then A must be positive semidefinite.
12. **(4 points)** Write a program that visualizes level sets of $f(x,y)$ (or do it manually). Draw them for (a) $f(x_1,x_2) = 4x_1^2 + x_1 x_2 + \frac{1}{10}x_2^2$; (b) $f(x_1,x_2) = \frac{x_2^2}{x_1}$, for $x_1, x_2 > 0$, (c) $f(x_1,x_2) = (1-x_1)^2 + 100(x_2 - x_1^2)^2$. The level set of a function f is the set $\{\mathbf{x}\colon f(\mathbf{x}) = c\}$ for all possible $c \in \mathbb{R}$. You only need to include plots, not a program.
13. **(3 points)** Show that the following functions are L-smooth: (a) $f(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - \mathbf{b}\|^2$; (b) $f(\mathbf{x}) = \sqrt{1 + \|\mathbf{x}\|^2}$; (c) $f(x) = x^3$ for $x \in [-3, 3]$.