LINKÖPING UNIVERSITY
Department of Mathematics
Anders Björn 2018-04-17

Exercises for MAI0065 *Functional Analysis*, Spring 2018

All exercises should be solved individually. You are allowed to discuss the exercises with the other students in the class, but you are not allowed to copy their solution nor any solution from anywhere else; you have to write your own solution. Moreover, you have to make sure that you understand all steps in your solution including all the logic used.

Your solutions of the following exercises should be presented at the problem seminars.

A1–**A**3
E 1.2:7,9 (In 9: Prove "second countable $\Rightarrow$ separable" for arbitrary topological spaces.)
E 1.3:3
A4 **A**5
E 1.4:3,13,14 (In 13 last part: it is enough to show for open bounded and closed bounded intervals.)
E 1.5:1,2
E 1.6:1,2
A6–**A**8

E 2.1:3,5–7,10
E 2.2:1,2
A9–**A**13
E 2.3:2,4,5,7,8 (In 2, you don't need to show that $\ell^\infty = (\ell^1)^*$.)
E 2.4:4,5,7,8
A14
E 2.5:3
A15 **A**16

E 3.1:7–11,13
E 3.2:1–5,10,11,15 (In 11 it is enough to show uniform convergence for $\|T\| < \delta$ for each $\delta < 1$. But, can you show it for all $\|T\| < 1$? The hint to 11 is useful, but seems to contain some error(s).)
A17–**A**18
E 4.1:10 (You may assume that X is a Hilbert space.)
A19–**A**20
E 3.3:3–7

Additional exercises

A1. In a metric space X, we define the balls $B(x,r) = \{y \in X : d(x,y) < r\}$ and $\widehat{B}(x,r) = \{y \in X : d(x,y) \le r\}$, where $x \in X$ and $r > 0$.
(a) Show that $B(x,r)$ is open.
(b) Show that $\widehat{B}(x,r)$ is closed.
(c) Give an example of a metric space X and x, r such that $\overline{B(x,r)} \neq \widehat{B}(x,r)$.
(d) Give and example of an unbounded metric space X and x, r such that $\widehat{B}(x,r)$ is open.

A2. Let
$$A = \left\{ (-1)^n + \frac{2}{n} : n = 1, 2, 3, ... \right\}$$
in the metric space $\mathbf{R}$.
(a) Find A's limit points and isolated points, and determine $\overline{A}$. Is A open? Is it closed?
(b) Let instead $X = \{x \in \mathbf{R} : -1 < x < 4\}$ and answer the same questions.
(c) Let instead $X = A$ and answer the same questions.
(In all cases X is equipped with the usual distance inherited from $\mathbf{R}$.)

A3. Let A and B be two subsets of a metric space.
  (a) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
  (b) Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
  (c) Give an example showing that it is possible to have $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
  (d) Let x be a nonisolated limit point A. Show that every neighbourhood of x contains infinitely many points of A.
  (e) Assume now instead that A is a subset of a topological space and that x is a nonisolated limit point of A. Is it then always true that every neighbourhood of x contains infinitely many points of A. Prove this, or give a counterexample.

A4. Recall the definitions of continuous and open functions (1.4.1 and 1.4.4) and Proposition 1.6.7.
  (a) Find a continuous function $f : \mathbf{R} \to \mathbf{R}$ which is not open.
  (b) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $F \subset \mathbf{R}$ be closed. Is it then true that $f(F)$ is closed? (Give proof or counterexample.)
  (c) Let $f : X \to Y$ be continuous and $F \subset X$ be closed, where X and Y are metric spaces, Is it then true that $f(F)$ is closed? (Give proof or counterexample.)

A5. Let $f : X \to Y$, where X and Y are metric spaces.
  (a) Show that f is continuous if and only if $f^{-1}(F)$ is closed for all closed $F \subset Y$.
  (b) Now assume that X and Y are toplogical spaces. Does the result in (a) hold? (Give proof or counterexample.) If not, which is (are) the most general assumption(s) *you* need to impose on X and/or Y for your proof to hold.
  (c) Show that f is continuous at $x \in X$ if and only if it is *sequentially continuous* at x, i.e. $f(x_j) \to f(x)$ whenever $(x_j)_{j=1}^{\infty}$ is a sequence with limit x.
  (d) Now assume that X and Y are toplogical spaces. Does the result in (c) hold? (Give proof or counterexample.) If not, which is (are) the most general assumption(s) *you* need to impose on X and/or Y for your proof to hold.

A6. Let $\{K_j\}_{j=1}^{\infty}$ be a decreasing sequence of nonempty compact sets in some Hausdorff space.
  (a) Show that $\bigcap_{j=1}^{\infty} K_j \neq \emptyset$.
  (b) Give a counterexample for the corresponding result for closed sets.

A7. A metric space X is *proper* if every set which is closed and bounded is also compact.
  (a) Show that if a metric space is proper then it is complete.
  (b) Give an example of a locally compact complete proper metric space.
  (c) Give an example of a complete nonproper metric space which is not locally compact.
  (d) Give an example of a locally compact noncomplete nonproper metric space.
  (e) Can you give an example of a locally compact complete nonproper metric space?

A8. Let (X, d) be a metric space. For $x \in X$ and a nonempty set $E \subset X$ define

$$\operatorname{dist}(x, E) = \inf\{d(x, y) : y \in E\}.$$

  (a) Show that if E is closed and $\operatorname{dist}(x, E) = 0$, then $x \in E$.
  (b) Use the dist function to show that X is a normal topological space.

A9. A *Hamel basis* for a vector space V is a family of vectors B such that
  (i) each finite subset of B is linearly independent;
  (ii) every vector in V can be written as a finite linear combination of vectors in B.

Show that every vector space has a Hamel basis.

A10. Let V be an infinite-dimensional vector space (i.e. it has an infinite Hamel basis). Give V two nonequivalent norms.

A11. Show that a Hamel basis of an infinite-dimensional Banach space cannot be countable. (Hint: Use Baire's category theorem.)

A12. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Assume that $Y \subset X$ (as sets), and that there is $A > 0$ such that $\|x\|_X \le A\|x\|_Y$ for $x \in Y$.

(a) If $(x_j)_{j=1}^\infty$ is a sequence converging to x in Y (in particular this implicitly requires $x_j \in Y$), what can you say about convergence in X?

(b) Let $T \in B(X)$ and assume that $T(Y) \subset Y$. Show that $T|_Y \in B(Y)$. (Hint: Use the closed graph theorem.)

A13. Let X be a Banach space of dimension at least 2 (it may be infinite-dimensional). Show that $\mathbf{B}(X)$ is noncommutative.

A14. (This is a continuation of **E 2.4**:7.)

(a) Show that 0 is in the weak closure of $A = \{x \in \ell^1 : \|x\| = 1\}$. (Hint: Give a basis of $\mathcal{O}(0)$ with respect to the weak topology, and use it in a suitable way.)

(b) Is there a countable subset $E \subset A$ such that 0 is in the weak closure of E?

A15. Assume that $x_n \to x$ weakly in a Banach space X.

(a) Show that $\sup_n \|x_n\| < \infty$.

(b) Show that $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.

(c) Give an example (in some Banach space, e.g. ℓ^2) where $\|x_n\| = 1$ for all n, but $\|x\| = 0$.

A16. Consider the sequence spaces ℓ^p.

(a) Let $1 < p < \infty$. Show that there is a weakly convergent sequence in ℓ^p which is not strongly (i.e. norm) convergent. How does this compare to exercise **E 2.4**:7?

(b) Find a bounded sequence in ℓ^1 which does not have a weakly convergent subsequence. Is it possible to find such a sequence if we replace ℓ^1 by ℓ^p, $1 < p < \infty$?

A17. Let H be a separable infinite-dimensional Hilbert space. Given an example of an unbounded operator which is everywhere defined. (Hint: Use a Hamel basis.)

A18. Let $A \subset B$, $A \neq B$, be unbounded operators. Show the following:

(a) If B is selfadjoint, then A is symmetric, but not selfadjoint.

(b) If A is selfadjoint, then B is not symmetric.

A19. Let H and K be Hilbert spaces and $W \in \mathbf{B}(H, K)$ be unitary. Assume that $A \in \mathbf{B}(K)$ and $B \in \mathbf{B}(H)$ satisfy $WB = AW$. Show that $\mathrm{sp}(A) = \mathrm{sp}(B)$ and $\mathrm{sp}_p(A) = \mathrm{sp}_p(B)$. Does your proof work for Banach spaces?

A20. Let H be a complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^\infty$. Let $\{a_n\}_{n=1}^\infty$ be a sequence with $\lim_{n\to\infty} a_n = 0$. Define the weighted shift operator A by $Ae_n = a_n e_{n+1}$.

(a) What is $\mathrm{sp}(A)$?

(b) Describe how $\mathrm{sp}_p(A)$ depends on $\{a_n\}_{n=1}^\infty$.