MAI0074: HOME ASSIGNMENT PROBLEMS

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1. (i) Let Ω be a set, and let $\{\mathcal{F}_i; i \in \Gamma\}$ be σ-algebras of subsets of Ω, where Γ is an arbitrary (possibly uncountable) index set. Prove, by checking all the conditions in the definition, that $\cap_{i\in\Gamma}\mathcal{F}_i$ is a σ-algebra.

   (ii) Let Ω be a set, and let $\mathcal{C}$ be a collection of subsets in Ω. Prove that there exists a σ-algebra $\mathcal{F}$ such that $\mathcal{C} \subset \mathcal{F}$, and such that $\mathcal{F} \subset \mathcal{G}$ for any other σ-algebra $\mathcal{G}$ such that $\mathcal{C} \subset \mathcal{G}$. [The σ-algebra $\mathcal{F}$ is called the σ-algebra generated by $\mathcal{C}$, and is denoted by $\sigma(\mathcal{C})$.]

2. Let $\mathcal{S}$ be the semiring of bounded half-open intervals $\mathcal{S} = \{(a, b] \subset \mathbb{R}; -\infty < a \leq b < \infty\}$. Prove that the σ-algebra generated by $\mathcal{S}$ is the same as the σ-algebra generated by the collection of bounded open intervals $\mathcal{C} = \{(a, b) \subset \mathbb{R}; -\infty < a \leq b < \infty\}$, and also the same as the σ-algebra generated by the collection of open sets in $\mathbb{R}$. (Please recall that the collection of open sets in $\mathbb{R}$ is not identical to $\mathcal{C}$.)

3. Let $\Omega = (0, 1]$, let $\mathcal{S}_0 = \{(a, b] \subset \mathbb{R}; 0 \leq a \leq b \leq 1\}$, and let $\mathcal{A}_0$ be the collection of finite disjoint unions of sets in $\mathcal{S}_0$. That is: $A \in \mathcal{A}_0$ if and only if $A = \cup_{i=1}^{n} B_i$, where $n \in \mathbb{N}$, and $\{B_i \in \mathcal{S}_0; i = 1, \ldots, n\}$ are disjoint sets.

   (i) Prove that $\mathcal{S}_0$ is a semiring, and that $\mathcal{A}_0$ is an algebra but not a σ-algebra.

   (ii) Define the set function $\mu : \mathcal{A}_0 \to \mathbb{R}_+$ as follows: $\mu(A) = 1$ if there exists an $\varepsilon_A > 0$ (depending on A) such that $(\frac{1}{2}, \frac{1}{2} + \varepsilon_A] \subset A$, and $\mu(A) = 0$ otherwise. Prove that μ is finitely, but not countably, additive.

4. Let $X : \Omega \to \mathbb{R}$ be a real valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Recall the definition of $\sigma(X)$ (the σ-algebra generated by X).

(i) Prove that X is $\mathcal{G}/\mathcal{R}$-measurable if and only if $\sigma(X) \subset \mathcal{G}$.

(ii) Prove that if $\mathcal{G} = \{\emptyset, \Omega\}$, then X is $\mathcal{G}/\mathcal{R}$-measurable if and only if X is constant.

(iii) Assume that $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{G}$. Prove that if X is $\mathcal{G}/\mathcal{R}$-measurable, then $\mathbb{P}(X = c) = 1$ for some constant $c \in \mathbb{R}$.

5. Let X and Y be real valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that Y is $\sigma(X)$-measurable if and only if there exists a $\mathcal{R}/\mathcal{R}$-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f \circ X$. Hint: For the "only if" part, prove the claim first for Y a simple function, then for Y nonnegative, then for a general Y.

6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f : \Omega \to \mathbb{R}_+$ be integrable. Define the set function $\nu : \mathcal{F} \to \mathbb{R}_+$ by

$$\nu(A) = \int_A f d\mu = \int I_A f d\mu \qquad \forall A \in \mathcal{F}.$$

(i) Prove that ν is a measure.

(ii) Prove that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mu(A) < \delta$, then $\nu(A) < \varepsilon$.

7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $\{f_n : \Omega \to \mathbb{R}; n = 1, 2, \ldots\}$ be integrable functions such that $f_n \uparrow f$, meaning (as usual) that $\{f_n; n = 1, 2, \ldots\}$ is a non-decreasing sequence which converges pointwise to a function $f : \Omega \to \mathbb{R}$. Prove that if $\sup_{n \geq 1} \int f_n d\mu < \infty$, then f is integrable and $\int f_n d\mu \to \int f d\mu$ as $n \to \infty$. (Please note that we are not assuming $\{f_n; n = 1, 2, \ldots\}$ to be nonnegative.)

8. A function $f : (a, b] \to \mathbb{R}$ on a bounded interval $(a, b] \subset \mathbb{R}$ is said to be Riemann integrable, with Riemann integral r, if the following condition holds: for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| r - \sum_{i=1}^{n} f(x_i)\nu(I_i) \right| < \varepsilon,$$

for any finite partitioning of $(a, b]$ into disjoint subintervals $\{I_i; i = 1, \ldots, n\}$ satisfying $\max_{i=1,\ldots,n} \nu(I_i) < \delta$, and any set of real numbers $\{x_i \in I_i; i = 1, \ldots, n\}$. (Here, ν is the Lebesgue measure.)

Assume that f is Borel measurable and bounded. Prove that if f is Riemann integrable, the Riemann integral coincides with the Lebesgue integral.

9. Let $\{X_i; i = 1, 2, 3\}$ be independent random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 0.5$, $i = 1, 2, 3$. Let $X_4 = X_1 X_2 X_3$. Prove that any three of the four random variables $\{X_1, X_2, X_3, X_4\}$ are independent, but that the four random variables $\{X_1, X_2, X_3, X_4\}$ are not independent.

10. Let X be a real valued random variable with cumulative distribution function $F_X : \mathbb{R} \to [0, 1]$. Recall that $F_X(x) = \mathbb{P}(X \leq x)$ for each $x \in \mathbb{R}$.

    (i) Let $c \in \mathbb{R}$. Prove that (the Lebesgue integral) $\int_{-\infty}^{\infty} (F_X(x + c) - F_X(x))dx = c$.

    (ii) Assume that F_X is a continuous function. Prove that $\mathbb{E}(F_X(X)) = 0.5$.

11. Let $\{X_t; t = 1, 2, \ldots\}$ be a countably infinite sequence of random variables, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $\{X_t; t = 1, 2, \ldots\}$ are independent if and only if the n-dimensional random variable $(X_1, \ldots, X_n)^T$ is independent of X_{n+1} for each $n = 1, 2, \ldots$ (meaning that $\sigma(X_1, \ldots, X_n)$ is independent of $\sigma(X_{n+1})$ for each $n = 1, 2, \ldots$).

12. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let the events $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, be independent. Let $S_n = \sum_{i=1}^{n} I_{A_i}$ for each $n = 1, 2, \ldots$. Prove that $\frac{S_n}{n} - \frac{1}{n} \sum_{i=1}^{n} P(A_i)$ converges to 0 in probability as $n \to \infty$.

13. Let $\{X_t; t = 1, 2, \ldots\}$ be i.i.d. random variables, such that

$$P(X_1 = (-1)^k k) = \begin{cases} \frac{a}{k^2 \ln k} & \text{for } k = 2, 3, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

    where $a = \left(\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}\right)^{-1}$. Let $S_n = \sum_{i=1}^{n} X_i$ for each $n = 1, 2, \ldots$.

    (i) Show that $E(|X_1|) = \infty$.

    (ii) Show that there is a finite constant γ such that $\frac{S_n}{n} \xrightarrow{p} \gamma$.

14. Let $\{X_t; t = 1, 2, \ldots\}$ be independent random variables, such that $X_n \xrightarrow{p} 0$. Does it follow that $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} 0$? Prove, or disprove by means of a counterexample.

15. Let $\{X_n; n = 1, 2, \ldots\}$ be iid random variables, such that $P(X_1 = 2^k) = 2^{-k}$ for each $k = 1, 2, \ldots$. Prove that $P(\limsup_{n \to \infty} \frac{X_n}{n \ln n} = \infty) = 1$.

16. Let $\{X_i; i = 1, 2, \ldots\}$ be independent (not necessarily identically distributed) random variables, and let $S_n = \sum_{i=1}^n X_i$ for each $n = 1, 2, \ldots$. Prove that if $E(X_i) = 0$ for each $i = 1, 2, \ldots$, and $E(X_i^4) \leq C$ for each $i = 1, 2, \ldots$, where $C \in (0, \infty)$ is a constant, then $\frac{S_n}{n}$ converges to 0 almost surely as $n \to \infty$.

17. Let $\{X_i; i = 1, 2, \ldots\}$ be independent (not necessarily identically distributed) random variables. Define the random power series B by $B(s) = \sum_{k=0}^{\infty} X_k s^k$.

    (i) Prove that the convergence radius of B is, with probability 1, equal to a *deterministic* constant $r \in [0, \infty]$.

    (ii) Assume that $\{X_i; i = 1, 2, \ldots\}$ are iid random variables such that $\mathbb{P}(X_1 \neq 0) > 0$. Prove that the constant r is either 1 or 0, depending on whether the expectation $\mathbb{E}(\ln^+ |X_1|)$ is finite or infinite. (Here, $\ln^+ x = \ln(\max(x, 1))$.)

18. Let $\{X_n; n = 1, 2, \ldots\}$ and $\{Y_n; n = 1, 2, \ldots\}$ be random variables. Assume that $X_n \to X$ in distribution as $n \to \infty$, where X is a random variable, and that $Y_n \to a$ in distribution as $n \to \infty$, where $a \in \mathbb{R}$ is a constant. Prove that $X_n + Y_n \to X + a$ in distribution as $n \to \infty$.

19. Let $\{\mu_n; n = 1, 2, \ldots\}$ be probability measures on $(\mathbb{R}, \mathcal{R})$. Let $f : \mathbb{R} \to [0, \infty)$ be a nonnegative measurable function such that $f(x) \to \infty$ as $x \to \pm\infty$. Prove that if

$$\sup_{n \geq 1} \int f(x) d\mu_n(x) < \infty,$$

then $\{\mu_n; n = 1, 2, \ldots\}$ is tight, meaning that for each $\varepsilon > 0$, we can find an $0 < M < \infty$ such that

$$\sup_{n \geq 1} \mu_n([-M, M]^c) < \varepsilon.$$

20. Let $\{X_n; n = 1, 2, \ldots\}$ and X be random variables, such that $X_n \xrightarrow{d} X$ as $n \to \infty$. It is well known that the family of probability distributions $\{\mu_{X_n}; n = 1, 2, \ldots\}$ is tight, meaning that for each $\varepsilon > 0$, we can find an $0 < M < \infty$ such that

$$\sup_{n \geq 1} \mu_{X_n}([-M, M]^c) < \varepsilon.$$

(i) Prove that the family of characteristic functions $\{\varphi_{X_n}; n = 1, 2, \ldots\}$ is equicontinuous, meaning that for each $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$\sup_{n\geq 1}\left|\varphi_{X_n}(t+h) - \varphi_{X_n}(t)\right| < \varepsilon \qquad \forall |h| < \delta, t \in \mathbb{R}.$$

(ii) Let $-\infty < a < b < \infty$. Prove that

$$\sup_{a\leq t\leq b}\left|\varphi_{X_n}(t) - \varphi_X(t)\right| \to 0, \qquad \text{as } n \to \infty.$$

21. Let $\{X_n; n = 1, 2, \ldots\}$ and $\{Y_n; n = 1, 2, \ldots\}$ be random variables, such that X_n and Y_n are independent for each $n = 1, 2, \ldots$. Show that if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ as $n \to \infty$, then $X_n + Y_n \xrightarrow{d} X + Y$ as $n \to \infty$.

22. Give an example of a sequence of random variables $\{X_n; n = 1, 2, \ldots\}$ which converges in distribution to a standard normal random variable as $n \to \infty$, but which also has the property that that $E(|X_n|^k) = \infty$ for each positive integer $k = 1, 2, \ldots$.

23. Let $\{X_n; n = 1, 2, \ldots\}$ be independent random variables such that $E(X_n) = 0$ for each $n = 1, 2, \ldots$, $E(X_n^2) = 1$ for each $n = 1, 2, \ldots$, and $E(|X_n|^{2+\varepsilon}) \leq M$ for each $n = 1, 2, \ldots$, where $\varepsilon > 0$ and $0 < M < \infty$. Show that $\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ converges in distribution to a standard normal random variable as $n \to \infty$.

24. Let $\{X_n; n = 1, 2, \ldots\}$ be independent random variables such that $P(|X_n| \leq C) = 1$ for each $n = 1, 2, \ldots$, where $0 < C < \infty$, and $\sigma_n^2 = V(\sum_{i=1}^n X_i) \to \infty$ as $n \to \infty$. Show that $\frac{1}{\sqrt{\sigma_n^2}}(\sum_{i=1}^n X_i - E(\sum_{i=1}^n X_i))$ converges in distribution to a standard normal random variable as $n \to \infty$.

25. Let the random variable X have a stable distribution with characteristic function

$$\varphi_X(t) = e^{-b|t|^\alpha} \qquad \forall t \in \mathbb{R},$$

where $0 < \alpha < 2$ and $b > 0$. Prove that $E(|X|^p) < \infty$ for $p < \alpha$. Hint: Make use of the inequality

$$P(|X| > \frac{2}{a}) \leq \frac{1}{a}\int_{-a}^{a}(1 - \varphi_X(t))dt \qquad \forall a > 0.$$

26. Let the random variable X have an infinitely divisible distribution with characteristic function φ_X. Prove that

$$\varphi_X(t) \neq 0 \qquad \forall t \in \mathbb{R}.$$

27. Let X be a random variable, with probability distribution μ_X and characteristic function φ_X. Let $Y = (Y_1, \ldots, Y_d)^T$ be a d-dimensional (that is, $\mathbb{R}^d$-valued) random variable with characteristic function

$$\varphi_Y(t_1, \ldots, t_d) = \varphi_X(t_1 + \cdots + t_d) \qquad \forall (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

What is the probability distribution of Y?

28. Let μ and ν be σ-finite measures on a measurable space $(\Omega, \mathcal{F})$. Prove that the Lebesgue decomposition of ν with respect to μ is unique.

29. Let (X, Y) be a 2-dimensional (that is, $\mathbb{R}^2$-valued) random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the probability space $(\mathbb{R}^2, \mathcal{R}^2, \mu)$, where μ is the probability distribution of (X, Y). Assume that μ has a density f with respect to the Lebesgue measure on $(\mathbb{R}^2, \mathcal{R}^2)$. Define $\mathcal{G}$ as the σ-algebra $\{A \times \mathbb{R}; A \in \mathcal{R}\} \subset \mathcal{R}^2$.

    (i) Prove that a version of the conditional probability $\mu(\mathbb{R} \times B|\mathcal{G})$, where $B \in \mathcal{R}$, is given by

    $$\mu(\mathbb{R} \times B|\mathcal{G}) = \frac{\int_B f(x,u)du}{\int_{\mathbb{R}} f(x,u)du} \qquad \forall (x,y) \in \mathbb{R}^2.$$

    (ii) Prove that a version of the conditional probability $P(Y \in B|\sigma(X))$, where $B \in \mathcal{R}$, is given by

    $$P(Y \in B|\sigma(X)) = \frac{\int_B f(X(\omega),u)du}{\int_{\mathbb{R}} f(X(\omega),u)du} \qquad \forall \omega \in \Omega.$$

30. Let X be an integrable random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $Y = E(X|\mathcal{G})$, where $\mathcal{G} \subset \mathcal{F}$. Prove that if $E(Y^2) = E(X^2) < \infty$, then $P(X = Y) = 1$.