# MAI0090: HOME ASSIGNMENT PROBLEMS 

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1. Let $\left\{X_{i} ; i=1,2, \ldots\right\}$ be independent random variables such that $E\left(X_{i}\right)=0$ and $E\left(X_{i}^{2}\right)=\sigma_{i}^{2} \in(0, \infty)$ for $i=1,2, \ldots$ Let $S_{0}=0$, and $S_{n}=\sum_{i=1}^{n} X_{i}$ for $n=1,2, \ldots$. Prove that $\left\{S_{n}^{2}-\sum_{i=1}^{n} \sigma_{i}^{2} ; n=0,1, \ldots\right\}$ is a martingale with respect to the filtration $\left\{\mathcal{F}_{n} ; n=0,1, \ldots\right\}$, where $\mathcal{F}_{0}=\{\emptyset, \Omega\}$ and $\mathcal{F}_{n}=\sigma\left(X_{1}, X_{2}, \ldots, X_{n}\right)$.
2. Let $\left\{X_{n} ; n=1,2, \ldots\right\}$ and $\left\{Y_{n} ; n=1,2, \ldots\right\}$ be supermartingales with respect to the filtration $\left\{\mathcal{F}_{n} ; n=1,2, \ldots\right\}$. Let $N$ be a stopping time with respect to $\left\{\mathcal{F}_{n} ; n=1,2, \ldots\right\}$ such that $Y_{N} \leq X_{N}$.
(i) Prove that $\left\{U_{n} ; n=1,2, \ldots\right\}$, defined by

$$
U_{n}=X_{n} I\{N>n\}+Y_{n} I\{N \leq n\} \quad \forall n=1,2, \ldots
$$

is a supermartingale with respect to $\left\{\mathcal{F}_{n} ; n=1,2, \ldots\right\}$.
(ii) Prove that $\left\{V_{n} ; n=1,2, \ldots\right\}$, defined by

$$
V_{n}=X_{n} I\{N \geq n\}+Y_{n} I\{N<n\} \quad \forall n=1,2, \ldots
$$

is a supermartingale with respect to $\left\{\mathcal{F}_{n} ; n=1,2, \ldots\right\}$.
3. Let $\left\{X_{i} ; i=1,2, \ldots\right\}$ be a martingale, and assume that there exists a constant $C \in(0, \infty)$ such that either $P\left(\sup _{i=1,2, \ldots} X_{i} \leq C\right)=1$ or $P\left(\inf _{i=1,2, \ldots} X_{i} \geq-C\right)=1$. Prove that $\sup _{i=1,2, \ldots} E\left(\left|X_{i}\right|\right)<\infty$.
4. Let $\left\{X_{i} ; i=1,2, \ldots\right\}$ be independent random variables such that $E\left(X_{i}\right)=0$ and $E\left(X_{i}^{2}\right)=\sigma_{i}^{2} \in(0, \infty)$ for each $i=1,2, \ldots$, and assume in addition that there is a constant $C \in(0, \infty)$ such that $\left|X_{i}\right| \leq C$ for each $i=1,2, \ldots$. Let $S_{0}=0$, and $S_{n}=\sum_{i=1}^{n} X_{i}$ for $n=1,2, \ldots$. Use the result proven in Problem 1, and the optional stopping theorem for bounded stopping times, to prove that

$$
P\left(\max _{i=1, \ldots, n}\left|S_{n}\right| \leq x\right) \leq \frac{(x+C)^{2}}{V\left(S_{n}\right)} \quad \forall x>0
$$

5. Let $\left\{X_{i} ; i=0,1, \ldots\right\}$ be a martingale with respect to a filtration $\left\{\mathcal{F}_{n} ; n=0,1, \ldots\right\}$, and let $Z_{n}=X_{n}-X_{n-1}$ for $n=1,2, \ldots$ Prove that if $E\left(X_{0}^{2}\right)<\infty$ and $\sum_{i=1}^{\infty} E\left(Z_{i}^{2}\right)<\infty$, then $\left\{X_{i} ; i=0,1, \ldots\right\}$ converges a.s. and in $L^{2}$ norm as $n \rightarrow \infty$.
6. $\left\{X_{i} ; i=1,2, \ldots\right\}$ be nonnegative random variables, and let $D=$ $\cup_{i=1}^{\infty}\left\{X_{i}=0\right\}$. Assume that $P\left(D \mid X_{1}, \ldots, X_{n}\right) \geq \delta(x)>0 P$-a.s. on the set $\left\{X_{n} \leq x\right\}$, for each $x \geq 0$. Use Lévy's 0-1 law to prove that $P\left(D \cup\left\{\lim _{n \rightarrow \infty} X_{n}=\infty\right\}\right)=1$.
Hint: Show that $P\left(D \cap\left\{\lim _{n \rightarrow \infty} X_{n}=\infty\right\}^{c}\right)=P\left(\left\{\lim _{n \rightarrow \infty} X_{n}=\right.\right.$ $\infty\}^{c}$ ).
7. Let $\left\{X_{i} ; i=0,-1,-2, \ldots\right\}$ be a backwards martingale with respect to a filtration $\left\{\mathcal{F}_{i} ; i=0,-1,-2, \ldots\right\}$. Prove that if $E\left(\left|X_{0}\right|^{p}\right)<\infty$ for some $1<p<\infty$, then $\left\{X_{i} ; i=0,-1,-2, \ldots\right\}$ converges in $L^{p}$ norm as $n \rightarrow-\infty$ (in addition to converging a.s and in $L^{1}$ norm) to a limit $X_{-\infty}$.
8. Let $S_{0}=0$, and let $S_{n}=\sum_{i=1}^{n} X_{i}$ for each $n=1,2, \ldots$, where $\left\{X_{i} ; i=\right.$ $0,1, \ldots\}$ are iid random variables such that $P\left(X_{1}=1\right)=P\left(X_{1}=\right.$ $-1)=\frac{1}{2}$. Then, $\left\{S_{n}^{2}-n ; n=0,1,2, \ldots\right\}$ is a martingale with respect to the filtration $\left\{\sigma\left(X_{0}, \ldots, X_{n}\right) ; n=0,1, \ldots\right\}$ (se Problem 1). Let $T=$ $\inf \left\{n>0 ;\left|S_{n}\right| \geq c\right\}$, where $c$ is a positive integer.
(i) Prove that $E(T)=c^{2}$.
(ii) Determine constants $a$ and $b$ such that $\left\{W_{n} ; n=0,1,2, \ldots\right\}$, defined by $W_{n}=S_{n}^{4}-6 n S_{n}^{2}+b n^{2}+a n$ for each $n=0,1,2, \ldots$, is a martingale with respect to the filtration $\left\{\sigma\left(X_{0}, \ldots, X_{n}\right) ; n=\right.$ $0,1, \ldots\}$, and use this to compute $E\left(T^{2}\right)$.
9. Let $S_{0}=0$, and let $S_{n}=\sum_{i=1}^{n} X_{i}$ for each $n=1,2, \ldots$, where $\left\{X_{i} ; i=0,1, \ldots\right\}$ are iid random variables such that $P\left(X_{1}=1\right)=$ $P\left(X_{1}=-1\right)=\frac{1}{2}$. Let $Y_{n}=\max _{i=0, \ldots, n} S_{i}$ for $n=0,1, \ldots$. Show that $\left\{Y_{i} ; i=0,1, \ldots\right\}$ is not a Markov chain with respect to the filtration $\left\{\sigma\left(X_{0}, \ldots, X_{n}\right) ; n=0,1, \ldots\right\}$.
Hint: $\left\{Y_{i} ; i=0,1, \ldots\right\}$ is a Markov chain with respect to the filtration $\left\{\sigma\left(X_{0}, \ldots, X_{n}\right) ; n=0,1, \ldots\right\}$, if and only if

$$
P\left(Y_{n+1}=y_{n+1} \mid \cap_{i=1}^{n}\left\{X_{i}=x_{i}\right\}\right)=P\left(Y_{n+1}=y_{n+1} \mid Y_{n}=y_{n}\right),
$$

for all $x_{1}, \ldots, x_{n} \in\{-1,1\}, s_{n}=\sum_{i=1}^{n} x_{i}$, and $y_{n}=\max _{i=0, \ldots, n} s_{i}$.
10. Two urns (urn $A$ and urn $B$, respectively) contain $b$ balls each. The balls are either red ( $r$ balls) or green ( $g$ balls; $r+g=2 b$ ). At time $t=1,2, \ldots$, one ball is chosen at random from each urn and moved to the other urn. Let $X_{t}$ be the number of red balls in urn $A$ immediately after the draw at time $t$. Then, $\left\{X_{t} ; t=0,1, \ldots\right\}$ is a Markov chain (you need not prove this). Compute the transition probability, and decide whether the states of the chain are recurrent or transient.
11. Let $\left\{X_{i} ; i=0,1, \ldots\right\}$ be a Markov chain. Let $A \in \sigma\left(X_{0}, \ldots, X_{n}\right)$ and $B \in \sigma\left(X_{n}, X_{n+1}, \ldots\right)$. Prove that $P\left(A \cap B \mid X_{n}\right)=P\left(A \mid X_{n}\right) P\left(B \mid X_{n}\right)$.
Hint: Assume that the chain is defined on $\left(S^{\mathbb{Z}_{+}}, \sigma(\mathcal{C})\right)$, and use the (generalized) Markov property.
12. Let $\left\{X_{i} ; i=0,1, \ldots\right\}$ be a Markov chain on a finite or countably infinite state space $S$. Prove that $\rho_{x, z} \geq \rho_{x, y} \rho_{y, z}$ for any $x, y, z \in S$.
13. Let $\left\{X_{i} ; i=0,1, \ldots\right\}$ be an irreducible Markov chain on a finite or countably infinite state space $S$.
(i) Prove that if the function $f: S \rightarrow \mathbb{R}$ satisfies

$$
\begin{equation*}
f(x) \geq \sum_{y \in S} p(x, y) f(y) \quad \forall x \in S \tag{*}
\end{equation*}
$$

then $\left\{f\left(X_{n}\right) ; n=0,1, \ldots\right\}$ is a supermartingale with respect to the filtration $\left\{\sigma\left(X_{0}, \ldots, X_{n}\right) ; n=0,1, \ldots\right\}$.
(ii) Prove that $\left\{X_{i} ; i=0,1, \ldots\right\}$ is recurrent if and only if any nonnegative function $f: S \rightarrow \mathbb{R}_{+}$satisfying (*) must be constant.
14. Let $\left\{X_{i} ; i=0,1, \ldots\right\}$ be an irreducible and positive recurrent Markov chain on a finite or countably infinite state space $S$. Let $T_{y}=\inf \{n \geq$ $\left.1 ; X_{n}=y\right\}$. Prove that $E_{x}\left(T_{y}\right)<\infty$ for any $x, y \in S$.
15. A chessboard consists of $8 \times 8=64$ squares. A knight is placed in one of the corners. There are no other pieces on the board. At each time $t=1,2, \ldots$, the knight is moved at random to one of the squares to which it is legally permitted to move, with equal probability for all such squares. Compute the expected number of moves it takes for the knight to return to its initial position. (Note that a knight is only permitted to move by taking two steps in one direction followed by one step in a perpendicular direction.)
16. Let $\left\{X_{i} ; i=0,1, \ldots\right\}$ be an irreducible and aperiodic Markov chain on a finite state space $S$. Assume that $\left\{X_{i} ; i=0,1, \ldots\right\}$ is irreducible and aperiodic. Prove that there is an $n_{0} \geq 1$ such that, if $n \geq n_{0}$, $p^{n}(x, y)=P_{x}\left(X_{n}=y\right)>0$ for all $x, y \in S$.
17. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\varphi: \Omega \rightarrow \Omega$ be a measurable mapping. Let $\mathcal{I}_{s}=\left\{A \in \mathcal{F} ; A=\varphi^{-1}(A)\right\}$ (the sets that are invariant in the strict sense).
(i) Prove that $\mathcal{I}_{s}$ is a $\sigma$-algebra.
(ii) Prove that a random variable $X$ is $\mathcal{I}_{s}$-measurable if and only if $X=X \circ \varphi$.
18. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let the mapping $\varphi: \Omega \rightarrow \Omega$ be measure preserving under $P$. Let $\mathcal{I}_{s}=\left\{A \in \mathcal{F} ; A=\varphi^{-1}(A)\right\}$, and let $\mathcal{I}=\left\{A \in \mathcal{F} ; P\left(A \Delta \varphi^{-1}(A)\right)=0\right\}$, where $A \Delta B=\left(A \cap B^{c}\right) \cup\left(B \cap A^{c}\right)$.
(i) Let $A \in \mathcal{F}$, and let $B=\cup_{k=0}^{\infty} \varphi^{-k}(A)$. Prove that $\varphi^{-1}(B) \subset B$.
(ii) Let $B \in \mathcal{F}$ be such that $\varphi^{-1}(B) \subset B$, and let $C=\cap_{k=0}^{\infty} \varphi^{-k}(B)$. Prove that $\varphi^{-1}(C)=C$.
(iii) Prove that $A \in \mathcal{I}$ if and only if there exists a $C \in \mathcal{I}_{s}$ such that $P(A \Delta C)=0$.
19. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\varphi: \Omega \rightarrow \Omega$ be a measurable mapping. If two probability measures $P_{1}$ and $P_{2}$ are preserved by $\varphi$, then so is $P=\alpha P_{1}+(1-\alpha) P_{2}$, for any $\alpha \in[0,1]$ (please check!). Prove that $\varphi$ is ergodic under the probability measure $P$ if and only if $P$ cannot be expressed as $P=\alpha P_{1}+(1-\alpha) P_{2}$, where $P_{1}$ and $P_{2}$ are distinct probability measures preserved by $\varphi$, and $\alpha \in(0,1)$.
20. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let the mapping $\varphi: \Omega \rightarrow \Omega$ be measure preserving under $P$. Let $\left\{X_{n} ; n=1,2, \ldots\right\}$ and $X$ be random variables such that $X_{n} \rightarrow X$ a.s., and such that $E\left(\sup _{n=1,2, \ldots}\left|X_{n}\right|\right)<$ $\infty$. Prove that

$$
\frac{1}{n} \sum_{k=0}^{n-1} X_{k} \circ \varphi^{k} \rightarrow E(X \mid \mathcal{I}) \quad P \text {-a.s., as } n \rightarrow \infty
$$

21. Let $\Omega_{0}$ be the space of functions $\omega:[0, \infty) \rightarrow \mathbb{R}$, and let $\mathcal{F}_{0}$ be the $\sigma$-algebra generated by the collection of finite-dimensional sets $\left\{\cap_{i=1}^{n}\left\{\omega\left(t_{i}\right) \in A_{i}\right\} ; 0 \leq t_{1}<t_{2}<\cdots<t_{n} ; A_{i} \in \mathcal{R}, i=1,2, \ldots, n ; n=\right.$
$1,2, \ldots\}$. Moreover, let $\left(\mathbb{R}^{\mathbb{Z}_{+}}, \sigma(\mathcal{C})\right)$ be the space of real valued sequences equipped with the $\sigma$-algebra generated by the collection of finite-dimensional sets $\left\{\cap_{i=0}^{n}\left\{x_{i} \in A_{i}\right\} ; A_{i} \in \mathcal{R}, i=1,2, \ldots, n ; n=\right.$ $1,2, \ldots\}$. Prove that a set $B \subset \Omega_{0}$ belongs to $\mathcal{F}_{0}$ if and only if there is a sequence $0 \leq t_{1}<t_{2}<\ldots$, and a set $C \in \sigma(\mathcal{C})$, such that $B=\left\{\omega \in \Omega_{0} ;\left(\omega\left(t_{1}\right), \omega\left(t_{2}\right), \ldots\right) \in C\right\}$.

Hint: Use Dynkin's $\pi-\lambda$ theorem.
22. Let $\{B(t) ; t \geq 0\}$ be a Brownian motion. Fix $t>0$, and define $\Delta_{k, n}=$ $B\left(t k 2^{-n}\right)-B\left(t(k-1) 2^{-n}\right)$ for $k=1, \ldots, 2^{n}$ and $n=0,1, \ldots$.
(i) Compute $E\left(\left(\sum_{k=1}^{2^{n}} \Delta_{k, n}^{2}-t\right)^{2}\right)$ for $n=0,1, \ldots$
(ii) Use the Borel-Cantelli lemma to prove that $\sum_{k=1}^{2^{n}} \Delta_{k, n}^{2} \rightarrow t$ a.s. as $n \rightarrow \infty$.
23. Let $\{B(t) ; t \geq 0\}$ be a Brownian motion. Let $T^{0}=\inf \{t>0 ; B(t)=0\}$ and $R^{0}=\inf \{t>1 ; B(t)=0\}$. Prove that

$$
P_{x}(R>1+t)=\int P_{y}\left(T^{0}>t\right) p_{1}(x, y) d y \quad \forall t>0 .
$$

24. Let $\{B(t) ; t \geq 0\}$ be a Brownian motion. Prove that, for each $s_{0}>0$, with probability 1 there exists a sequence $\left\{t_{n} ; n=1,2, \ldots\right\}$ decreasing to $s_{0}$, such that each $t_{n}$ is a local maximum of $\{B(t) ; t \geq 0\}$.
Hint: Use the fact that $X(t)=B\left(t+s_{0}\right)-B\left(s_{0}\right), t \geq 0$, is a Brownian motion starting at 0 ,
