Matematiska institutionen, Linköpings universitet / Göran Bergqvist

Matrix Analysis, 2020

Hand-in exercises, sheet 1

To be solved individually and handed in no later than Monday 2/11, 2020. Complete solutions must be presented. Write your name on each page.

1. Suppose that A is $n \times n$ and non-singular, and that B and C are $n \times m$. Use Schur complements to prove that $\det(A + BC^T) = \det(A)\det(I_m + C^T A^{-1} B)$.
How does the formula simplify if $m = 1$? What is $\det(I_n + \mathbf{u}\mathbf{v}^T)$, where $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$?
Demonstrate the formula for a non-trivial example in the case $n = 3$ and $m = 1$.

2. Suppose that A, B, C and D are $n \times n$ matrices. Is it true in general that $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} A & C \\ B & D \end{pmatrix}$?
Or that $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} D & C \\ B & A \end{pmatrix}$? Give proofs or counterexamples.

3. Prove that circulant matrices are normal, and diagonalize some 4×4 circulant matrix (choose 4 distinct elements). What is the diagonalizing matrix? Does it depend on your elements?

4. Suppose that A is not normal. What is the minimum possible rank of $A^*A - AA^*$?
Hint: what is its trace?

5. Suppose the real 12×12 matrix A has Jordan form (block diagonal notation)

$$J = \begin{pmatrix} 1+2i & 1 \\ 0 & 1+2i \end{pmatrix} \oplus (1+2i) \oplus \begin{pmatrix} 1-2i & 1 \\ 0 & 1-2i \end{pmatrix} \oplus (1-2i) \oplus \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \oplus (-1)$$

Find the real Jordan form, the Weyr form, the rational form, and the rational canonical forms over $\mathbb{C}$ and over $\mathbb{R}$ of A.

6. If J is a single Jordan block corresponding to the eigenvalue $\lambda = -1$, show that J^k is similar to J for any odd integer $k \geq 3$. If $\lambda = -1$ is the only eigenvalue of a matrix A, show that A^k is similar to A (k odd).

7. Suppose that A is real $n \times n$ and tridiagonal such that $a_{i,i+1}a_{i+1,i} > 0$, $i = 1, 2, \ldots, n-1$. Show that there is a real diagonal matrix D such that DAD^{-1} is symmetric. What can be said about the eigenvalues of A? Show the calculation for some 4×4 non-symmetric matrix A (find D).

8. Let A be $n \times n$ and Hermitian, $a \in \mathbb{R}$, $y \in \mathbb{C}^n$, $B = \begin{pmatrix} A & y \\ y^* & a \end{pmatrix}$, and $C = A \pm yy^*$.
Show that rank(B) - rank(A) can take only the values 0, 1, or 2, and give examples for all three possibilities. Show that rank(C) - rank(A) can take only the values -1, 0, or 1.

9. Show that replacing a non-zero element in a matrix by 0 can increase its spectral norm. Find matrices A and B such that $||A||_2 > ||B||_2$ and $|||A|||_2 < |||B|||_2$ ($||\cdot||_2$ and $|||\cdot|||_2$ are Frobenius and spectral norms, respectively).

10. If A and B are $n \times n$ matrices, show that $||AB||_2 \neq ||BA||_2$ in general. Then show that
$||AB||_2 \leq |||A|||_2||B||_2$, $||AB||_2 \leq ||A||_2|||B|||_2$, and $||A||_2 \leq \sqrt{n}|||A|||_2$.