Due Date: This assignment is due on Friday, 15 April, 2016, by 5pm to jan.nordstrom@liu.se *and* per.pettersson@uib.no. You are strongly encouraged to use LaTeX.

Problem 1: Monte Carlo methods

a) Monte Carlo estimators

The *bias* of an estimator $\hat{S}$ of a QOI denoted S is defined as $E(\hat{S}) - S = 0$. Are the following estimators of $Var(X)$ unbiased (i.e. the bias is 0)? If not, what is the bias?

i) $\frac{1}{N}\sum_{n=1}^{N}(X_n - \bar{X})^2$

ii) $\frac{1}{N-1}\sum_{n=1}^{N}(X_n - \bar{X})^2$

where $\bar{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$.

b) Accelerated sampling: MLMC

Prove that the error in the multilevel Monte Carlo mean estimator is given by

$$\|E(u_L) - E_{MLMC}(u_L)\|^2_{L^2(\Omega,L^2(D))} = \sum_{l=1}^{L}\frac{\sigma_l^2}{N_l}$$

where $\sigma_l^2 = \|E(u_l - u_{l-1}) - (u_l - u_{l-1})\|^2_{L^2(\Omega,L^2(D))}$ and N_l is the number of samples per level. Explain why it is expected to be more efficient than standard Monte Carlo sampling.

Problem 2: Modeling input parameters with KL expansions

Assume that you want to solve a problem where your input parameter(s) can be represented by a random field Y through the Karhunen-Loeve expansion,

$$Y(x) = \bar{Y}(x) + \sum_{n=1}^{\infty}\sqrt{\lambda_n}\phi_n(x)\xi_n.$$

The KL expansion must be truncated due a finite number of terms (N), but you want to capture at least 95% of the input variance.

a) Derive an expression for the fraction of total variance captured by the N-term truncated KL expansion as a function of λ_n, ϕ_n, N.

b) Choose a covariance function and either compute the eigenpairs $\{\phi_n, \lambda_n\}$ or find analytical expressions in the literature. (Hint: Gaussian and exponential covariance kernels have analytically derived eigenpairs to be found in journal papers and textbooks on subsurface flow and UQ.) Choose a spatial domain size (single dimension) and plot the sum of the eigenvalues λ as a function of the number of KL terms retained. For a domain of length D (you choose D), do this for correlation parameters being equal to D and $D/2$, respectively.

c) Consider the covariance kernel $C(x, x') = \delta(x - x')$ (white noise) where δ denotes the Dirac distribution. Derive the corresponding eigenvectors and eigenfunctions. What do you find? What can you say about the KL expansion of this fully uncorrelated process?

Problem 3: gPC representation

a) Generalized polynomial chaos formulations of UQ problems typically start from infinite series expansions, ending up with a formulation involving a finite number of gPC terms. This truncation introduces a stochastic truncation error that propagates in subsequent operations on the gPC series. Verify that the finite order expansion of the product of f and g is different from the product of the expansions of f and g.

b) We will consider the problem of computing the gPC coefficients of a given function when analytical expressions are not available. Consider ξ bounded uniformly in $[-1, 1]$. Compute the coefficients of the gPC expansions of ξ^3 and $\sin(\xi)$ for order $M = 1$, $M = 3$, and $M = 5$ using the least-squares approach with different choices of N , i.e., $N = 10$, $N = 100$, $N = 1000$. Select the realizations ξ_j as a set of points distributed randomly in the interval $[-1, 1]$.

c) Instead of Monte Carlo integration for computation of the gPC coefficients, one may choose the points in random space according to a numerical integration rule. For the previous problem use pseudospectral projection with different choices of N , i.e., $N = M$, $N = 1.2M$, $N = 2M$. Select Gauss-Legendre quadrature.

Problem 4: PROPAGATION OF UNCERTAINTY USING GPC
Consider the wave equation in one spatial dimension,

$$\frac{\partial^2 u}{\partial t^2} - c(\xi)\frac{\partial^2 u}{\partial x^2} = 0$$

a) Choose an initial condition and determine the analytical solution as a function of x, t, ξ. What are your boundary conditions?
[Hint: introduce a second solution variable and write it as a system of two linear advection equations. You have seen the solutions for advection equations during class.]

b) Let $\xi \sim U(-1, 1)$ and introduce suitable generalized polynomial chaos. Express gPC coefficient u_k using your exact solution and state the resulting expression. Choose a time t and plot the first few gPC coefficients as a function of x. You might need to use numerical integration.

c) Perform a stochastic Galerkin projection of the PDE, initial and boundary conditions.

d) Solve the resulting stochastic Galerkin system numerically for at least two different orders of gPC expansion. Compare with the analytical solution and plot the result.

Problem 5: WELL-POSEDNESS AND STABILITY OF GPC RELATED PROBLEMS
Consider the advection-diffusion problem (modelling the Navier-Stokes equations) posed on the spatial domain $0 \leq x \leq 1$

$$\begin{aligned} u_t + au_x + bu &= \epsilon u_{xx} + F \\ cu + du_x &= g_0, \quad x = 0 \\ u &= g_1, \quad x = 1 \end{aligned} \tag{1}$$

augmented with an initial condition f. In (1), we have random coefficients $a = a(\xi) \neq 0$, $b = b(\xi) \neq 0$, $c = c(\xi) \neq 0$, $d = d(\xi) \neq 0$, $\epsilon = \epsilon(\xi) > 0$ where ξ is a random variable.

a) Use the energy method and choose signs of a,b,c,d such that (1) is strongly well posed. You can use $F = g_1 = 0$ in this task. Hint: The signs of the coefficients are related, they will depend on each other.

b) Discretize (1) using SBP-SAT in space and show strong stability using the discrete energy method. You can use $F = g_1 = 0$ in this task. Hint: mimic the continuos estimate derived in **a)**.

c) Write a code that solves (1) using the 2nd order SBP-SAT in space and RK4 in time. Use the method of manufactured solution (with $w = (1 - e^{a(x-1)/\epsilon})/(1 - e^{-a/\epsilon}) + e^{-3t} sin(8\pi(x - at)))$ to get the data for $F, g_, g_1$ and f. Use $a = 1$, $b = 0$ and $\epsilon = 1, 0.1, 0.01$. Choose your c and d appropriately. Vary your mesh size and show second order convergence. Plot the solution as a function of x for $t = 0, 1, 2, 3, 5, 10$.

Next, expand the solution using gPC such that $u \approx \sum_{i=0}^{i=M} u_i(x, t)\psi_(\xi)$ and such that (1) goes over to

$$\begin{aligned} U_t + AU_x + BU &= EU_{xx} \\ CU + DU_x &= G, \quad x = 0 \\ U &= 0, \quad x = 1 \end{aligned} \tag{2}$$

where $U = (u_0, u_1 ... u_M)^T$ and we have used $F = g_1 = 0$.

d) Show that the sign you have chosen on the coefficients in **a)** to make (1) strongly well posed is preserved in the definiteness of the matrices by the gPC expansion. Hint: do not expand the coefficients is a gPC expansion.

e) Use the energy method and show that also (2) lead to a strongly well posed problem. Hint: follow/mimic the scalar procedure in **a)**.