Due Date: This assignment is due on Monday, 22 February, 2016. The purpose of this assignment is to review essential background material *before* the lectures start. We have not given you any reading material for this, but the exercises are written in a format to help you identify key words in case you need to search in textbooks (or Wikipedia) for relevant definitions. Good luck!

Problem 1:
Let X be a standard Gaussian (i.e. normal with zero mean and unit variance) random variable and let $Y = X^2$.
a) Are X and Y correlated?
b) Are X and Y dependent?
c) Let $Z \sim N(\mu, \sigma^2)$ (normal random variable with mean μ and variance σ^2). Compute the expectation and variance of the lognormal random variable $V = \exp(Z)$.
d) What is the distribution of the random variable $1/V$?

Problem 2:
The random variables $u \sim U(0, 1)$ and $v \sim U[2, 3]$ are independent uniform random variables. Compute the expectations (mean values) $E(u)$, $E(v)$, $E(uv)$ and the variances $Var(u)$, $Var(v)$ and $Var(uv)$.

Problem 3:
In uncertainty quantification we often update probabilites based on new information as a result of new data being available over time. Studies have shown that this is very non-intuitive for the human brain (for an overview, see Kahneman 2011). So let's make sure we know how to do this and consider the following situation. A disease is present in 0.5% of a population. If the disease is present, a blood test correctly indicates its presence in 99% of the cases of an individual carrying the disease. The same blood test also indicates presence of the disease in 5% of the cases where the tested individual does not have the disease. If the test result tells a person that (s)he has the disease, what is the probability that this is wrong?
Hint: if you don't know where to start, read about conditional probability and Bayes' theorem.

Problem 4:
Knowledge of PDEs, how to discretize them and the use of probability theory results in "UQ for PDEs". The continuous problem (the PDE together with boundary conditions (BC) and initial conditions) under consideration must be well-posed for a meaningful UQ assessment. A continuous problem is well-posed if: i) the solution <u>exists</u> (the correct number and position of the BC), ii) is <u>bounded by the data</u> (correct form of BC) and iii) is <u>unique</u>. In this task we will investigate ii) and to some extent its dependence on i).

Consider the domain $0 \le x \le 1$ for $t \ge 0$ and the parabolic and hyperbolic problems

1. $u_t + \epsilon u_{xx} = 0$ for $u(0, t) = u(1, t) = 0$ and the three cases $\epsilon = -1, 0, +1$.

2. $u_t + au_x = 0$ for $u(0, t) = 0$ and the three cases $a = -1, 0, +1$.

Which of the 6 cases above lead to a solution bounded by data (and existence)? Motivate your answer. Hint: Define the energy $||u||^2 = \int_0^1 u^2 dx$, multiply the equations above with u and integrate over the domain.

Problem 5:
We discretize the problem 2 above and consider $\vec{U} = (U_0, U_1, ..., U_{N-1}, U_N)^T$ as an approximation to u in the grid-points $(0, \Delta x, 2\Delta x, .., (N-1)\Delta x, 1)$. Let D be the $(N+1) \times (N+1)$ differentiation matrix such that $D\vec{\phi} \approx \vec{\phi}_x$, where $\vec{\phi}$ is the smooth function ϕ injected in the grid-points. Furthermore, let the $(N+1) \times (N+1)$ matrix $P = P^T > 0$ define a norm and a quadrature formula such that $\vec{\phi}^T P \vec{\phi} \approx \int_0^1 \phi^2 dx$. The two matrices are connected through the summation-by-parts (SBP) relation $PD + (PD)^T = B = diag(-1, 0, 0, ..., 0, 0, +1)$.

Show that the scheme

3. $\vec{U}_t + aD\vec{U} = -(a/2)P^{-1}U_0\vec{e}_0$

where $\vec{e}_0 = (1, 0....0)$ leads to an energy estimate (similar to the one you should aim for in **Problem 4**, task 2 above). For which value of a do we have a stable scheme?

Hint: Define the energy $||\vec{U}||_P^2 = \vec{U}^T P \vec{U}$, multiply equation 3 with $\vec{U}^T P$ from the left and use the SBP matrix relation above.