Stochastic Galerkin Methods for PDEs

Graduate Course, Spring 2016, Linköping University

Examination

Work in small groups (or individually) on a mini project. Project topics will be provided by the lecturers. The deliverables include a report (5 pages) and mandatory homework.

DAY 1: Basic Concepts

(Estimated time: 4×45 minutes)

Introduction: Why uncertainty quantification? General problem definition: PDE with stochastic IC, BC or parameters

- [P] (2×45 minutes) Representation of random fields via spectral expansions:
  - Karhunen-Loeve decomposition based on covariance statistics, optimal representation
  - Generalized polynomials chaos, orthogonal polynomials, L2 convergence
  - Localized representations for non-smooth functions
  - Multiple dimensions: stochastic tensor grids, sparse grids, Smolyak quadrature rules
- [J] (45 minutes) PDE Theory
  - Elliptic/hyperbolic/parabolic PDEs
  - Well-posedness and boundary conditions
  - Linear vs nonlinear PDEs
- [P] (45 minutes) Example: gPC formulation for simple advection problem
  - Stochastic Galerkin: coupled system to be solved once - new problem
  - Stochastic Collocation: Few samples of original problem (move to day 3?)
  - Least-squares/regression: Samples dependent on order of gPC (move to day 3?)

Reading Material: TPME11.

DAY 2: Linear Problems

(Estimated time: 4 × 45 minutes)

- [J] (45 minutes) Hyperbolic Problems ($u_t + au_x = 0$)
  - Discretization and semi-discrete analysis of SG advection problem
- [P] (45 minutes) Parabolic Problems ($u_t = \epsilon u_{xx}$)
  - Stochastic Galerkin formulation and analysis
- [J] (45 minutes) Parabolic Problems ($u_t = \epsilon u_{xx}$)
  - Discretization and discrete analysis
- [P] (45 minutes) Elliptic Problems ($(ku_x)_x = f$)
  - Demonstration of very efficient gPC representation e.g. in comparison to standard Monte Carlo
  - Sparsity of gPC basis for elliptic problems

  This topic is also suitable for introducing multiple stochastic dimensions (if not covered during day 1).

Reading Material: XK02, GX08

DAY 3: Non-intrusive Methods

(Estimated time: 3 × 45 minutes)

- [J/P] (45 minutes) Summary of the material from day 1-2
- [P] (2 × 45 minutes) Non-intrusive methods
  - Stochastic collocation
  - Least-squares/regression
  - Compressive sensing: how to beat Nyquist (if time permits)
  - Examples

Reading material: TPME11

DAY 4: Nonlinear Problems (Burgers' equation)

(Estimated time: 3×45 minutes)

- [J] ($1\text{-}2 \times 45$ minutes) Nonlinear analysis for stochastic problem guided by deterministic analysis
  - Well-posedness
  - Effect of lack of boundary data (analysis of 2×2-case)
- [P] ($1\text{-}2 \times 45$ minutes) Analysis of the exact solution of the stochastic Burgers' equation
  - Smoothness of the solution, SG approximation vs. original stochastic formulation
  - Multiple discontinuities in stochastic Galerkin systems
- [J & P] Introduction to project work/assignments

Reading material: PIN15 Ch. 6

DAY 5: Advanced topics

(Estimated time: 2 or 3×45 minutes)

- [J] Sensitivity of different PDEs
  - Advection-diffusion
- [P] Multiple stochastic dimensions
- [P] Alternative gPC basis functions: wavelets, spatially adaptive gPC

Literature

GX08 Gottlieb, Xiu, Galerkin Method for Wave Equations with Uncertain Coefficients, Commun. Comput. Phys., Vol. 3, No. 2, pp. 505-518, 2008.

PIN15 Pettersson, Iaccarino, Nordström, Polynomial Chaos Methods for Hyperbolic Partial Differential Equations, Springer, 2015.

TPME11 Tuminaro, Phipps, Miller, Elman, Assessment of Collocation and Galerkin Approaches to Linear Diffusion Equations with Random Data, International Journal for Uncertainty Quantification, Vol. 1, No. 1, pp. 19-33, 2011.

XK02 Xiu, Karniadakis, Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos, CMAME, Vol. 191, pp. 49274948, 2002.