Suggested topics for presentations, Optimal Transport

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Contents

1 A game-theoretic approach to duality

Present the proof of the following minimax theorem of Sion, which itself is a generalization of a theorem of Von Neumann in game theory in the Euclidean setting. (Alternatively state it as below but give the von Neumann proof only for compact subsets in $\mathbb{R}^n$ assuming the Brouwer fixed point theorem is known). Then proceed to prove the fundamental Kantorovich-Koopmans duality theorem below.

Theorem 1.1 (Sion)**.** *Let U be a compact convex subset of a topological vector space, and let V be a convex subset of a topological vector space. Let $f : U \times V \to \mathbb{R}$ satisfy*

(i) $f(x, \cdot)$ is continuous and concave on V for each $x \in U$,

(ii) $f(\cdot, y)$ is continuous and convex on U for each $y \in V$.

Then

$$\min_{x\in U} \sup_{y\in V} f(x, y) = \sup_{y\in V} \min_{x\in U} f(x, y).$$

Theorem 1.2 (Kantorovich-Koopmans duality)**.**

$$W_c(\mu, \eta) = \sup \left\{ -\int u d\mu - \int v d\eta : -(u, v) \in \mathrm{Lip}_c \right\}. \tag{1}$$

Suggested source: The course lecture notes section 4.1, Section 2.1 in [10].

2 The case $c(x,y) = \|x-y\|$ in subsets of $\mathbb{R}^n$

This is a demanding topic/presentation.

As was mentioned during the course the following is proved in [11] and [6], using a limiting argument as $p \searrow 1$.

Theorem 2.1. *There exists an optimal map G for the Monge minimization problem.*

This is actually just part of Theorem 3.1 [11]. The topic for this presentation is to introduce the concepts (mass transfer sets/rays) and explain what their Theorem 3.1 says in detail (proving the theorem is not reasonable).

3 Euclidean isoperimetric inequality/Sobolev inequality

Present a proof of the isoperimetric and the Sobolev inequality. (It is enough to do one in detail and then explain how the other can be proved by similar methods).

Theorem 3.1 (Isoperimetric inequality)**.** *If $\Omega \subset \mathbb{R}^n$ is a bounded subset with smooth boundary, then*

$$n\lambda(B(0,1))^{1/n}\lambda(\Omega)^{(n-1)/n} \leq \sigma(\partial\Omega),$$

where σ denotes the surface area measure on $\partial\Omega$.

Theorem 3.2 (Sobolev Inequality)**.**

$$\|f\|_{L^{p^*}(R^n)} \leq C(n,p)\|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in C_c^\infty(\mathbb{R}^n),$$

where $p^* = np/(n-p)$.

Suggested source: Section 4.2 and 4.3 of [2].

4 Brenier's polar factorization theorem

Theorem 4.1 (Brenier's polar factorization theorem)**.** *If $r : X \to X$ is a vector field such that $\lambda(r^{-1}(A)) = 0$ whenever $\lambda(A) = 0$, then*

$$r = Dv \circ \phi,$$

where v is convex and ϕ is measure preserving.

Also compare this to the Helmholtz decomposition. *Suggested source: Proposition 1.28, Remark 1.29 of [2].*

5 Wasserstein distance

Let $X = Y$ be a compact metric space with $d_X = d_Y = d$ and let $c(x,y) = d(x,y)^p/p$ Prove that

$$d_p(\mu,\eta) = W_c(\mu,\eta)^{1/p},$$

is a metric on the space $\mathcal{P}(X)$ of all (Borel) probability measures on X. This distance is often called the Wasserstein p-distance.

If time permits one can also consider mentioning some properties that is inherited by $(\mathcal{P}(X), d_p)$ from X, like geodesic properties etc. in particular in the case $p = 2$.

Suggested source: Section 2.1 (-2.3) of [2].

6 Finite spaces and the simplex method

A typical linear programming problem can be given as follows. Let C be a (real) row matrix of length N, B a column matrix of length N and A an $M \times N$-matrix. Minimize $C\Gamma$ subject to the constraints $A\Gamma = B$ and $\Gamma \geq 0$ (where the latter means that all entries in the column matrix Γ are non-negative).

Explain first how the problem in Example **??** can be put on this form. Present how such a problem can be solved in a mechanical way using e.g. the simplex method. Furthermore do an example e.g. with the following numbers:

$$\mu = \delta_{x_1} + 4\delta_{x_2}, \quad \eta = 2\delta_{y_1} + 3\delta_{y_2},$$

and

$$c(x_1, y_1) = 1, \quad c(x_1, y_2) = 2, \quad c(x_2, y_1) = 3, \quad c(x_2, y_2) = 3.$$

Suggested source: Appendix to [7] and e.g. the Wikipedia entry on the simplex method.

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