Homework 1

Homework 1.1 *Let* $S^2 = \{(x, y, x) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, *and let*

$$\begin{aligned} U_S &= \{(x, y, z) \in S^2 : z \neq 1\} \\ U_N &= \{(x, y, z) \in S^2 : z \neq -1\}, \end{aligned}$$

together with $\vec{x}_S : U_S \to \mathbb{R}^2$, $\vec{x}_N : U_N \to \mathbb{R}^2$ *defined by*

$$\begin{aligned} \vec{x}_S(x, y, z) &= \big(x_S(x, y, z), y_S(x, y, z)\big) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \\ \vec{x}_N(x, y, z) &= \big(x_N(x, y, z), y_N(x, y, z)\big) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right). \end{aligned}$$

(a) Show that $\vec{x}_S$ *is the stereographic projection from the north pole* $(0, 0, 1)$ *onto the* (x, y)*-plane.*

(b) Prove that $\mathcal{A} = \{(U_S, \vec{x}_S), (U_N, \vec{x}_N)\}$ *is a* C^∞*-atlas for* S^2.

Homework 1.2 *For* $\theta \in \mathbb{R}$, *define* $r_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ *as*

$$r_\theta(x, y, z) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z).$$

(a) Show that r_θ *restricts to a bijective map* $r_\theta : S^2 \to S^2$.

(b) Compute

$$\begin{aligned} &\vec{x}_S \circ r_\theta \circ \vec{x}_S^{-1} \qquad && \vec{x}_N \circ r_\theta \circ \vec{x}_N^{-1} \\ &\vec{x}_S \circ r_\theta \circ \vec{x}_N^{-1} \qquad && \vec{x}_N \circ r_\theta \circ \vec{x}_S^{-1}, \end{aligned}$$

and show that they are smooth maps on their respective domains. Given the fact that $r_\theta : S^2 \to S^2$ *is continuous, this proves, by Proposition 1.55, that* r_θ *is a* C^∞ *map.*

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