Homework 10

Homework 10.1 *Let (M, g) be a Riemannian manifold and let ∇ be an affine connection. Show that if ∇ satisfies Kozul's formula, i.e. if*

$$\begin{aligned} 2g(\nabla_X Y, Z) = X\big(g(Y,Z)\big) + Y\big(g(X,Z)\big) - Z\big(g(X,Y)\big) \\ - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, then ∇ is a metric and torsion-free connection. (Note that there is no need to introduce local coordinates to prove this fact.)

Homework 10.2 *Let (S^2, g) be the sphere together with the Riemannian metric*

$$g_p = \begin{cases} \frac{4}{(1+x_N^2+y_N^2)^2}\big(dx_N \otimes dx_N + dy_N \otimes dy_N\big) & \text{if } p \in U_N \\ 4\big(dx_S \otimes dx_S + dy_S \otimes dy_S\big) & \text{if } p = (0,0,-1) \end{cases}$$

(as defined in Homework 8.1).

(a) Compute the Christoffel symbols of the Levi-Civita connection on (S^2, g) in the chart $(U_N, \vec{x}_N)$.

(b) For a two-dimensional manifold, there is only one independent component of the Riemannian curvature tensor. Hence, to compute all curvature components, it is enough to determine one of them, since the others are given by using the symmetries. Compute the curvature component R_{1212} with respect to the chart $(U_N, \vec{x}_N)$.

(c) Show that the curve $c : [0, \pi] \to S^2$, given by

$$c_\alpha(t) = \big(\sin(\alpha)\cos(t), \cos(\alpha)\cos(t), \sin(t)\big)$$

for arbitrary $\alpha \in \mathbb{R}$, is a geodesic. How does this curve look like when $\alpha = 0$?

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