Homework 12

Homework 12.1 *Let* $f : \Sigma \to M$ *be an embedding of a manifold* Σ *into a Riemannian manifold* (M, g) *(with Levi-Civita connection* $\bar{\nabla}$*), and let* $\mathcal{P} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ *denote the pointwise orthogonal projection onto* $T_p\Sigma \subseteq T_pM$*. Setting* $\Pi = \mathrm{id} - \mathcal{P}$ *(where* $\mathrm{id} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ *denotes the identity map) one defines*

$$\nabla_X Y = \mathcal{P}(\bar{\nabla}_X Y)$$
$$\alpha(X, Y) = \Pi(\bar{\nabla}_X Y)$$

for $X, Y \in \mathfrak{X}(\Sigma) \subseteq \mathfrak{X}(M)$*.*

(a) *Show that* ∇ *is an affine connection on* Σ*.*

(b) *Show that* $\alpha(X, Y) = \alpha(Y, X)$ *for all* $X, Y \in \mathfrak{X}(\Sigma)$*. (Hint: Use the fact that* $\nabla, \bar{\nabla}$ *are Levi-Civita connections.)*

Homework 12.2 *The sphere (of radius* $r > 0$*) can be parametrized by spherical coordinates:*

$$\vec{x}(\theta, \varphi) = \big(r \cos\varphi \sin\theta, r \sin\varphi \sin\theta, r \cos\theta\big)$$

where $\theta \in (0, \pi)$ *and* $\varphi \in (0, 2\pi)$*.*

(a) *Show that* $\vec{x}'_\theta(\theta, \varphi)$ *and* $\vec{x}'_\varphi(\theta, \varphi)$ *are linearly independent for all* θ *and* φ *in the given intervals.*

(b) *Compute the induced metric with respect to the parametrization* $\vec{x}(\theta, \varphi)$*.*

(c) *Compute the Gaussian curvature of the sphere from the definition*

$$K = \frac{\det II}{\det g},$$

where II *denotes the second fundamental form.*

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