Homework 4

Homework 4.1 *Let D_1 and D_2 be derivations of $C^\infty(M)$.*

(a) Show that

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

is a derivation of $C^\infty(M)$.

(b) Let $(U, \vec{x})$ be a chart on the manifold M. Show that if $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$ in $(U, \vec{x})$, then

$$[X, Y] = \left(X^k \frac{\partial}{\partial x^k} Y^i - Y^k \frac{\partial}{\partial x^k} X^i \right) \frac{\partial}{\partial x^i}.$$

(c) Show that $\mathfrak{X}(M)$ is a Lie algebra with respect to the Lie bracket $[\cdot, \cdot]$. That is, show that

1. $[X, Y] = -[Y, X]$,
2. $[X, aY + bZ] = a[X, Y] + b[X, Z]$
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$,

for $X, Y, Z \in \mathfrak{X}(M)$ and $a, b \in \mathbb{R}$.

Homework 4.2 *Let $(U, \vec{x})$ and $(V, \vec{y})$ be charts on the manifold M such that $U \cap V \neq \emptyset$. A vector field X may be written as*

$$X = X^i \frac{\partial}{\partial x^i} \qquad \text{and} \qquad X = \tilde{X}^i \frac{\partial}{\partial y^i}$$

in U and V, respectively, and it holds that $\tilde{X}^i = \frac{\partial y^i}{\partial x^k} X^k$.

(a) Let ω be a 1-form which, in local coordinates, can be written as

$$\omega = \omega_i dx^i \qquad \text{and} \qquad \omega = \tilde{\omega}_i dy^i,$$

in U and V, respectively. By using the transformation rule for vector fields above (and the fact that 1-forms are linear functionals of vector fields), show that

$$\tilde{\omega}_i = \frac{\partial x^k}{\partial y^i} \omega_k.$$

(b) Let $T \in T^1_1(TM)$ with local expressions

$$T = T^i_j \frac{\partial}{\partial x^i} \otimes dx^j \qquad \text{and} \qquad T = \tilde{T}^i_j \frac{\partial}{\partial y^i} \otimes dy^j$$

in U and V, respectively. Show that

$$\tilde{T}^i_j = \frac{\partial y^i}{\partial x^k} \frac{\partial x^l}{\partial y^j} T^k_l,$$

by using the transformation rules for vector fields and 1-forms.

(c) Show that the contractions of the two local expressions for T are equal; i.e., show that $T^i_i(p) = \tilde{T}^i_i(p)$ for all $p \in U \cap V$.

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