Homework 5

Homework 5.1 *Let $\omega \in \Omega^1(M)$ be a 1-form. The exterior derivative of ω is defined to be*

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$$

for $X, Y \in \mathfrak{X}(M)$. Show that, given a chart $(U, \vec{x})$, it follows that

$$d\omega = \left(\frac{\partial}{\partial x^i}\omega_k\right) dx^i \wedge dx^k$$

if the coordinate expression for ω is $\omega = \omega_i dx^i$.

Homework 5.2 *Let $M = \mathbb{R}^2 \backslash \{(0,0)\}$, for which an atlas can be provided by the single chart (M, id) (where $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the identity map). Moreover, let ω be the 1-form defined by*

$$\omega = \omega_x dx + \omega_y dy = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

(a) *Show that ω is a closed form; i.e that $d\omega = 0$.*

(b) *Compute*

$$\oint_C \omega := \int_0^{2\pi} \left(\omega_x\big(x(t), y(t)\big)\frac{dx}{dt} + \omega_y\big(x(t), y(t)\big)\frac{dy}{dt}\right)dt$$

where C is a circle, parametrized by $x(t) = \cos t$ and $y(t) = \sin t$.

(c) *For C, $x(t)$, $y(t)$ as in (b), show that*

$$\oint_C df := \int_0^{2\pi} \left(f'_x\big(x(t), y(t)\big)\frac{dx}{dt} + f'_y\big(x(t), y(t)\big)\frac{dy}{dt}\right)dt = 0,$$

for all $f \in C^\infty(M)$.

(d) *Use the results in (a)–(c) to show that $H^1(M) \neq 0$.*

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