Homework 7

Homework 7.1 *Let us define an action of* $\mathbb{Z}$ *on* $\mathbb{R}^2$ *as follows. For each* $n \in \mathbb{Z}$ *we define* $\rho_n : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$\rho_n(\alpha, t) = (\alpha + 2\pi n, (-1)^n t).$$

Furthermore, define a relation $\sim$ *on* $\mathbb{R}^2$ *by setting* $(\alpha, t) \sim (\alpha', t')$ *if there exists* $n \in \mathbb{Z}$ *such that* $\rho_n(\alpha, t) = (\alpha', t')$.

(a) *Show that* $\sim$ *is an equivalence relation.*

(b) *Set* $E = \mathbb{R}^2/\sim$ *and let* $p : \mathbb{R}^2 \to E$ *denote the projection onto the equivalence class in* E*; i.e.* $p(\alpha, t) = [(\alpha, t)]$. *Define the open sets*

$$V_1 = (0, 2\pi) \times \mathbb{R} \qquad V_2 = (-\pi, \pi) \times \mathbb{R}$$

and let p_i *denote the restriction of* p *to* V_i. *Setting* $U_i = p(V_i)$, *show that* $\{(U_1, \kappa_1), (U_2, \kappa_2)\}$ *is an atlas for* E, *where* $\kappa_i = p_i^{-1}$. *With this atlas, the set* E *becomes a smooth manifold.*

Homework 7.2 *Let* $E = \mathbb{R}^2/\sim$ *be as in Homework 7.1 and define the map* $\pi : E \to S^1$ *via*

$$\pi([(\alpha, t)]) = (\cos\alpha, \sin\alpha).$$

Show that $(E, \pi, S^1, \mathbb{R})$ *is a rank one vector bundle over* S^1. *This is the Möbius bundle, which is a non-trivial bundle over* S^1 *(i.e.* ***not*** *isomorphic to* $S^1 \times \mathbb{R}$*).*

Examiner: Joakim Arnlind `<joakim.arnlind@liu.se>`