Homework 8

Homework 8.1 *Let* $M = S^2$*, together with the atlas* $\mathcal{A} = \{(U_S, \vec{x}_S), (U_M, \vec{x}_N)\}$ *(cf. Homework 1.1). Furthermore, let* g *be the tensor field defined by*

$$g_p = \begin{cases} \frac{4}{(1+x_N^2+y_N^2)^2}\big(dx_N \otimes dx_N + dy_N \otimes dy_N\big) & \text{if } p \in U_N \\ 4\big(dx_S \otimes dx_S + dy_S \otimes dy_S\big) & \text{if } p = (0,0,-1). \end{cases}$$

(a) Compute the local expression for g *in the chart* $(U_S, \vec{x}_S)$*.*

(b) Show that (S^2, g) *is a Riemannian manifold.*

Homework 8.2 *Let* (S^2, g) *be as in Homework 8.1.*

(a) Compute the volume of (S^2, g)*.*

(b) Recall how the velocity vector of a curve $c : [a,b] \to M$ *is defined. Namely, if* $p \in M$ *and* $(U, \vec{x})$ *is a chart such that* $p \in U$ *and* $c(t_0) = p$*, then the velocity vector of* c *at* p *is defined as the tangent vector*

$$\dot{c}(t_0) = [(p, (v^1, \dots, v^n), (U, \vec{x}))],$$

where $v^i = \frac{d}{dt}|_{t_0} x^i(c(t))$*. On a Riemannian manifold* (M, g)*, the length of a curve is defined as*

$$L(c) = \int_a^b \sqrt{g(\dot{c}(t), \dot{c}(t))}dt.$$

Compute the length of the curve $c : [0, 2\pi] \to S^2$*, given by*

$$c(t) = \big(\cos t, 0, \sin t\big).$$

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