Homework 9

Homework 9.1 *Let ∇ be an affine connection on a manifold M, and let R be the corresponding curvature tensor.*

(a) Let $(U, \vec{x})$ and $(V, \vec{y})$ be charts on M such that $p \in U \cap V$. Derive a relation between the Christoffel symbols at p in the two different charts and conclude that the Christoffel symbols Γ^i_{jk} are not the components of a tensor.

(b) Show that R is multilinear over $C^\infty(M)$; i.e., show that

$$\begin{aligned} R(fX+Y,Z)W &= fR(X,Z)W + R(Y,Z)W \\ R(X,fY+Z)W &= R(X,Z)W + fR(X,Y)W \\ R(X,Y)(fZ+W) &= fR(X,Y)Z + R(X,Y)W. \end{aligned}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

Homework 9.2 *Let M be a manifold and let ∇ be a connection on M. A map $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called a $C^\infty(M)$-bilinear map if*

$$\begin{aligned} \alpha(fX+Y,Z) &= f\alpha(X,Z) + \alpha(Y,Z) \\ \alpha(X,fY+Z) &= f\alpha(X,Y) + \alpha(X,Z) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

(a) Let $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a $C^\infty(M)$-bilinear map. Show that $\bar{\nabla}$

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X,Y)$$

is a connection on M.

(b) Show that if ∇ and $\bar{\nabla}$ are connections on M then

$$\alpha(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$$

is a $C^\infty(M)$-bilinear map.

(c) Let $M = \mathbb{R}^2 \backslash \{(0,0)\}$, and let ∇ denote the connection on M defined by

$$\nabla_X Y = X^i \left(\frac{\partial}{\partial x^i} Y^k \right) \frac{\partial}{\partial x^k}$$

in the global chart $(\mathbb{R}^2 \backslash \{(0,0)\}, \mathrm{id})$, where we write $x^1 = x$ and $x^2 = y$. Show that for $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$, the map defined as

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{x^i}{x^2+y^2} \big(X^1 Y^1 + X^2 Y^2 \big) \frac{\partial}{\partial x^i}$$

is a connection on M and compute its Christoffel symbols Γ^i_{jk} as well as the curvature components R^i_{jkl}, for each $i, j, k \in \{1, 2\}$.

Examiner: Joakim Arnlind `<joakim.arnlind@liu.se>`