22m:033 Notes:
6.4 The Gram-Schmidt Process

Dennis Roseman
University of Iowa
Iowa City, IA

http://www.math.uiowa.edu/∼roseman

April 29, 2010

1 How to construct an orthogonal or orthonormal basis for given subspace.

Suppose we have a basis $\{\overrightarrow{x_1}, \ldots, \overrightarrow{x_p}\}$for a subspace W of R^n. We want a better basis $\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$—perhaps orthogonal, perhaps orthonormal. For the moment suppose we only want an orthogonal basis for W. Note that the dimension of W is p.

1.1 Dimension 1

Note that if W is one dimensional, our basis has one element $\overrightarrow{x_1}$. So we can simply let $\overrightarrow{v_1} = \overrightarrow{x_1}$

1.2 Dimension 2

If W is two dimensional, our basis has two elements $\{\overrightarrow{x_1}, \overrightarrow{x_2}\}$.

In this case we can

1. let $\overrightarrow{v_1} = \overrightarrow{x_1}$

2. let $\overrightarrow{v_2} = \overrightarrow{x_2} - \frac{\overrightarrow{x_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1}$

1.3 Dimension 3

If W is three dimensional, our basis has three elements $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}\}$.

In this case we can

1. let $\overrightarrow{v_1} = \overrightarrow{x_1}$
2. let $\overrightarrow{v_2} = \overrightarrow{x_2} - \frac{\overrightarrow{x_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1}$
3. let $\overrightarrow{v_3} = \overrightarrow{x_3} - \frac{\overrightarrow{x_3} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} - \frac{\overrightarrow{x_3} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} \overrightarrow{v_2}$

1.4 Dimension p

we continue this process until we get to

$$\overrightarrow{v_p} = \overrightarrow{x_p} - \frac{\overrightarrow{x_p} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} - \frac{\overrightarrow{x_p} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} \overrightarrow{v_2} - \cdots - \frac{\overrightarrow{x_p} \cdot \overrightarrow{v_{p-1}}}{\overrightarrow{v_{p-1}} \cdot \overrightarrow{v_{p-1}}} \overrightarrow{v_{p-1}}$$

Remark 1.1 In our process, we always guarantee that the first vector of our orthogonal basis is the same as the original basis. This is often significant in applications. We say that the Gram-Schmidt process starts with the first vector and completes a basis for W.

2 Orthonormal basis for W

Once we have an orthogonal basis $\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_p}\}$ for W we can easily obtain an orthonormal basis $\{\overrightarrow{u_1}, \ldots, \overrightarrow{u_p}\}$ by setting

$$\overrightarrow{u_i} = \frac{1}{||\overrightarrow{v_i}||}\overrightarrow{v_i}.$$

Example 2.1 Let

$$\overrightarrow{x_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \overrightarrow{x_2} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \overrightarrow{x_3} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Use the Gram-Schmidt process to transform the given basis to an orthonormal basis.

To begin, we let $\overrightarrow{v_1} = \overrightarrow{x_1}$.

Next

$$\overrightarrow{v_2} = \overrightarrow{x_2} - \frac{\overrightarrow{x_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix} - (-2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{x_3} - \frac{\overrightarrow{x_3} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} - \frac{\overrightarrow{x_3} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} \overrightarrow{v_2} =$$

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

So the orthogonal basis we get is

$$\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \}$$

To get an orthonormal basis we normalize each of these vectors and get:

$$\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \}$$

3 QR Factorization of matrices

We do not have time to cover this topic in this class.

4 Homework Problems

HINT: for the last three problems see pages 171-172 of text.

1. Let
$$\overrightarrow{x_1} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \overrightarrow{x_2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \overrightarrow{x_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$
These three vectors are a basis for a subspace W of R^4. Use the Gram-Schmidt process to transform the given basis to an orthonormal basis for W.

2. Use the Gram-Schmidt process to find an orthonormal basis for the null space of
$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

3. Use the Gram-Schmidt process to find an orthgonal basis for the column space of

$$\begin{pmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Use the Gram-Schmidt process to find an orthogonal basis for the column space of

$$\begin{pmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -3 & 5 & 1 \end{pmatrix}$$

   Hint: Compare this with the matrix from previous problem.