KORTFATTAT LÖSNINGSFÖRSLAG, DUGGA 1 20221014

91/92M413 TATB03

1. a) $\dfrac{\frac{x}{2}-\frac{8}{x}}{\frac{1}{4}+\frac{1}{x}} = \dfrac{\frac{x^2-16}{2x}}{\frac{4+x}{4x}} = \dfrac{(x-4)(x+4)}{2x(4+x)}\cdot 4x = 2(x-4)$ om $x \neq 0, -4$ ← SVAR:

b) $\displaystyle\sum_{k=0}^{19}\left(-\frac{1}{2}\right)^{2k+1} = -\frac{1}{2}\cdot\sum_{k=0}^{19}\left(\frac{1}{4}\right)^k = \Big\|$ geometrisk summa m kvot $\frac{1}{4}$, första term 1 & 20 termer $\Big\| = -\frac{1}{2}\,\dfrac{\left(\frac{1}{4}\right)^{20}-1}{\frac{1}{4}-1} = \underbrace{-\frac{1}{2}\left(-\frac{4}{3}\right)}_{=2/3}\dfrac{1-4^{20}}{4^{20}} = \dfrac{1-2^{40}}{3\cdot 2^{39}}$

SVAR: → $\dfrac{1-2^{40}}{3\cdot 2^{39}}$

c) $i z + 2\bar{z} = -\dfrac{1}{1+i} = -\dfrac{1\cdot(1-i)}{\underbrace{(1+i)(1-i)}_{=2}} = -\dfrac{1}{2}+\dfrac{i}{2}$ (*)

Sätt $z = a+ib$, $a,b\in\mathbb{R}$, så $\bar{z} = a-ib$. Då är (*) $\Leftrightarrow i(a+ib)+2(a-ib) = -\frac{1}{2}+\frac{i}{2} \Leftrightarrow$

$\Leftrightarrow ai - b + 2a - i2b = -\frac{1}{2}+\frac{i}{2} \Leftrightarrow \begin{cases} 2a-b = -1/2 & (\text{Re}) \\ a-2b = 1/2 & (\text{Im}) \end{cases} \Leftrightarrow a = b = -\frac{1}{2}$, dvs

$z = -\frac{1}{2}-\frac{i}{2}$ enda lösningen till (*) ← SVAR:

2. $\dfrac{2}{x}+1 \geq -\dfrac{3}{x-2} \Leftrightarrow \dfrac{2}{x}+1+\dfrac{3}{x-2} \geq 0 \Leftrightarrow \dfrac{2(x-2)+x(x-2)+3x}{x(x-2)} \geq 0 \Leftrightarrow$

$\Leftrightarrow \dfrac{2x-4+x^2-2x+3x}{x(x-2)} \geq 0 \Leftrightarrow \dfrac{x^2+3x-4}{x(x-2)} \geq 0 \Leftrightarrow \dfrac{(x-1)(x+4)}{x(x-2)} \geq 0$

Teckentabell:

		-4		0		1		2	
$(x-1)$	$-$		$-$		$-$	0	$+$		$+$
$(x+4)$	$-$	0	$+$		$+$		$+$		$+$
x	$-$		$-$	0	$+$		$+$		$+$
$(x-2)$	$-$		$-$		$-$		$-$	0	$+$
$\frac{(x-1)(x+4)}{x(x-2)}$	$+$	0	$-$	$\nexists$	$+$	0	$-$	$\nexists$	$+$

Teckentabellen visar att olikheten gäller om

SVAR: $\begin{cases} x \leq -4 \\ \text{eller} \\ 0 < x \leq 1 \\ \text{eller} \\ x > 2 \end{cases}$

3. $|x-3| + 2x = |2x+1| + 1$ (*) Då $|x-3| = \begin{cases} x-3 & \text{om } x-3 \geq 0 \Leftrightarrow x \geq 3 \\ -(x-3) & \text{om } x-3 \leq 3 \Leftrightarrow x \leq 3 \end{cases}$ och

$|2x+1| = \begin{cases} 2x+1 & \text{om } 2x+1 \geq 0 \Leftrightarrow x \geq -1/2 \\ -(2x+1) & \text{om } 2x+1 \leq 0 \Leftrightarrow x \leq -1/2 \end{cases}$ får vi tre fall: (I) $x \leq -1/2$, (II) $-1/2 \leq x \leq 3$ och (III) $x \geq 3$

Fall I: $x \leq -1/2$: (*) $\Leftrightarrow -(x-3)+2x = -(2x+1)+1 \Leftrightarrow 3x+3 = 0 \Leftrightarrow x = -1 \leq -1/2$: OK!

Fall II: $-1/2 \leq x \leq 3$: (*) $\Leftrightarrow -(x-3)+2x = (2x+1)+1 \Leftrightarrow x-1 = 0 \Leftrightarrow x = 1$ och $-1/2 \leq 1 \leq 3$ så OK!

Fall III: $x \geq 3$: (*) $\Leftrightarrow (x-3)+2x = (2x+1)+1 \Leftrightarrow x-5 = 0 \Leftrightarrow$

$\Leftrightarrow x = 5 \geq 3$: OK!

SVAR: $x = -1$, $x = 1$ & $x = 5$ löser (*)

4. $2z^2-3z-iz = 1-\frac{9}{2}i \iff z^2-\left(\frac{3}{2}+\frac{i}{2}\right)z = \frac{1}{2}-\frac{9}{4}i \iff$ // Kvad. komp // $\iff$

$\iff \left(z-\left(\frac{3}{4}+\frac{i}{4}\right)\right)^2 - \underbrace{\left(\frac{3}{4}+\frac{i}{4}\right)^2}_{=\frac{9}{16}+\frac{6i}{16}-\frac{1}{16}=\frac{1}{2}+\frac{3i}{8}} = \frac{1}{2}-\underbrace{\frac{9}{4}i}_{=\frac{18i}{8}} \iff \overbrace{\left(z-\left(\frac{3}{4}+\frac{i}{4}\right)\right)^2 = 1-\frac{15}{8}i}^{(*)}$

Sätt $z-\left(\frac{3}{4}+\frac{i}{4}\right) = a+ib$, $a,b\in\mathbb{R}$. Då fås att $(a+ib)^2 = 1-\frac{15}{8}i \iff$

$\iff a^2+2iab-b^2 = 1-\frac{15}{8}i$ och identifierar vi Re- & Im-del samt skaffar oss hjälpekv. via beloppet av $(*)$

Re: $a^2-b^2=1$ (1)

Im: $2ab = -\frac{15}{8}$ (2)

Abs: $a^2+b^2 = \sqrt{1^2-\left(\frac{15}{8}\right)^2} = \sqrt{\frac{289}{64}} = \frac{17}{8}$ (3)

(1)+(3) ger $2a^2 = 1+\frac{17}{8} \iff a^2 = \frac{25}{16} \iff a = \pm\frac{5}{4}$

(3)−(1) ger $2b^2 = \frac{17}{8}-1 \iff b^2 = \frac{9}{16} \iff b = \pm\frac{3}{4}$

(2) visar att a & b har olika tecken!

$\therefore$ $a = 5/4$ & $b = -3/4$ eller $a = -5/4$ & $b = 3/4$, vilket ger att

$z-\left(\frac{3}{4}+\frac{i}{4}\right) = \frac{5}{4}-\frac{3i}{4} \iff z = 2-\frac{i}{2}$ eller $z-\left(\frac{3}{4}+\frac{i}{4}\right) = -\frac{5}{4}+\frac{3i}{4} \iff z = -\frac{1}{2}+i$

SVAR

5. Strategi: Skriv om termerna i V.L. så att det går bryta ut $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-(k-1))!}$!

$$V.L. = \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{(n-1)!}{\underbrace{k!}_{=k\cdot(k-1)!}\underbrace{(n-1-k)!}_{=n-k-1}} - \frac{(n-1)!}{(k-1)!\underbrace{(n-1-(k-1))!}_{=(n-k)!}} =$$

$$\frac{n!}{(k-1)!\underbrace{(n-(k-1))!}_{=(n-k+1)!}}$$

$= \Big/\!\Big/$ "siktapå" $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$: förläng båda termerna med n, förläng 1:a termen med $(n-k)(n-k+1)$, 2:a termen med $(n-k+1)$ $\Big/\!\Big/ =$

$$= \frac{1}{n}\left(\frac{\overbrace{n\cdot(n-1)!}^{=n!}\cdot(n-k)(n-k+1)}{k\cdot(k-1)!\underbrace{(n-k-1)!(n-k)(n-k+1)}_{=(n-k+1)!}} - \frac{\overbrace{n\cdot(n-1)!}^{=n!}\cdot(n-k+1)}{(k-1)!\underbrace{(n-k)!\cdot(n-k+1)}_{=(n-k+1)!}}\right) =$$

$$= \frac{1}{n}\left(\frac{n!}{(k-1)!(n-k+1)!}\cdot\frac{(n-k)(n-k+1)}{k} - \frac{n!}{(k-1)!(n-k+1)!}\cdot(n-k+1)\right) =$$

$$= \binom{n}{k-1}\cdot\frac{(n-k+1)}{n}\cdot\left(\overbrace{\frac{n-k}{k}-1}^{=\frac{n-2k}{k}}\right) = \binom{n}{k-1}\cdot\frac{(n-k+1)(n-2k)}{nk} = \frac{2k^2-3kn-2k+n^2+n}{n\cdot k}\binom{n}{k-1} = H.L.$$

QED!