Notes on Riemann Surfaces

(Following David Singerman and Gareth A. Jones)

Universidad de Zaragoza

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Introduction

Riemann: Inverses of complex-analytical ~~functions~~ are (analytical) functions defined on surfaces: Riemann surfaces. Which functions are "analytical" determines the analytical structure of the surface

Riemann Surface X (Haussdorf, connected) with an open covering $X = \bigcup U_x$ s.t $\varphi_x : U_x \xrightarrow{\text{homeo}} V_x \overset{\text{open}}{\subseteq} \mathbb{C}$ and $\varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \longrightarrow \varphi_y(U_x \cap U_y)$ analytic

We will see that "analytic" functions $\Longleftrightarrow$ ramified covering

And the group of meromorphic functions of a Riemann surface ~~are~~ is (functionally) the group of fractional functions of a projective smooth curve (in that sense R.S $\Longleftrightarrow$ complex curve)

Finally any R.S is the quotient of either $\mathbb{S}^2$, $\mathbb{C}$ or $\mathbb{H}^2$ by a discrete subgroup of the automorphism group of $\mathbb{S}^2, \mathbb{C}, \mathbb{H}^2$ respectively.

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Chapter B (as in Background)

* Complex Numbers and Functions

i) Recall that a polynomial $f(z)$ of deg N always factorizes in linear factors

$$f(z) = \prod_{n_1 + \dots + n_r = N} (z - \alpha_i)^{n_i}$$

f has exactly N zero's counting multiplicity

ii) Most we will consider <u>domains</u> (or regions), i.e open, (path) connected sets in $\mathbb{C}$

iii) A function $f : D \longrightarrow \mathbb{C}$, D domain is differentiable at $z_0 \in D$ iff there is complex derivative at z_0. f is <u>analytic</u> on D if there is (complex) differentiable ~~at~~ each $z_0 \in D$.

Examples : Rational functions $f(z) = \dfrac{p(z)}{q(z)}$ are analytic on $D = \{ z \in \mathbb{C} \,/\, q(z) \neq 0 \}$

iv) If $f : D \longrightarrow \mathbb{C}$ is analytic and $f'(z_0) \neq 0$ then f is conformal at z_0

v) Cauchy - Riemann Equations $f : D \longrightarrow \mathbb{C}$ ($z \mapsto w$) given function. $f'(z_0)$ exists iff $f(x_0, y_0)$ (real) diff. and

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \qquad f(z) = f(x+iy) = w = u(x,y) + iv(x,y)$$

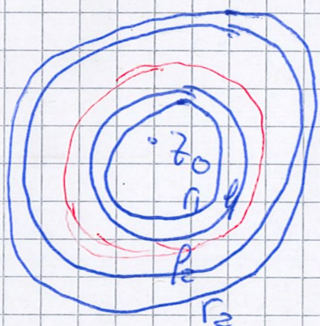
vi) Liouville's Th. If f in entire and there is a constant M s.t $|f(z)| < M$ $\forall z \in \mathbb{C}$ then f constant

viii) Let $0 \le r_1 < r_2$ (r_2 could be ∞), $z_0 \in \mathbb{C}$ and consider the region $D = \{ z \in \mathbb{C} \;\; r_1 < |z - z_0| < r_2 \}$.
Let f be analytic on D. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

both series converge absolutely on D and uniformly on sets $D_{\rho_1, \rho_2} = \{ z \mid \rho_1 \le |z - z_0| \le \rho_2 \}$ $r_1 < \rho_1 < \rho_2 < r_2$.

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi, \quad b_n = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - z_0)^{n-1} \, d\xi$$



γ circle of radius r
$r_1 < r < r_2$

If $r_1 = 0$ then f is analytic on a deleted neighbourhood of z_0 and z_0 is isolated singularity

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

if for an isolated singularity z_0 all but a finite number of b_n are zero, z_0 is a <u>pole</u> of f. If k is the highest number s.t $b_k \ne 0$, we say that z_0 is <u>a pole of order k</u>.

$$f(z) = \underbrace{\sum_{n=1}^{k} \frac{b_n}{(z - z_0)^n}}_{\text{principal part}} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

A function $f: D \longrightarrow \mathbb{C}$ that is analytic on D, except for poles in D is called meromorphic in D.

z_0 a pole of order k (near z_0) $f(z) = (z - z_0)^{-k} g(z)$, g analytic $\ne 0$

z_0 a zero of mult. k (near z_0) $f(z) = (z - z_0)^{k} h(z)$, $h \ne 0$ analytic

Theorem (Maximum-Modulus Principle) If $f(z)$ is a non-constant analytic function in a region D, then $|f(z)|$ has no maximum in D

* Topology (mainly Surfaces)

+ A surface X is a Haussdorff connected space s.t each pt $x \in X$ has an open neighbourhood U_x s.t U_x is homeomorphic to an open 2-disc V

Example $\hat{\mathbb{C}} \simeq S^2$ is a surface using the (inverse of) stereographic projection

$\varphi_N : \mathbb{C} \longrightarrow S^2 \setminus \{N\}$

$z \longmapsto \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$

$\psi_N : S^2 \setminus \{N\} \longrightarrow \mathbb{C}$ (pts not N)

$(x_1, x_2, x_3) \longrightarrow \frac{x_1 + i x_2}{1 - x_3}$

$\varphi_S : \mathbb{C} \longrightarrow S^2 \setminus \{S\}$

$z \longmapsto \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right)$

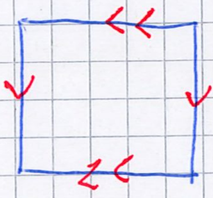
$\psi_S : S^2 \setminus \{S\} \longrightarrow \mathbb{C}$ (pts not S)

$(x_1, x_2, x_3) \longmapsto \frac{x_1 + i x_2}{1 + x_3}$

Observe that $\varphi_N^{-1} \varphi_S^{-1} (\mathbb{C} \setminus \{0\}) \longrightarrow \mathbb{C} \setminus \{0\}$, $z \mapsto \frac{1}{z}$ analytic

S^2 is a surface, we see it as $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

* Torus (i) any topological space homeomorphic to $S^1 \times S^1$
~~Any homeomorphic space to~~
The quotient space of the unit square $\{(x,y) \in \mathbb{R}^2 ; 0 \le x \le 1 ; 0 \le y \le 1\}$ with identifications $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$, $0 \le x \le 1$, $0 \le y \le 1$

 $\sim$  $\sim$ $\mathbb{C} / \Omega(0,1,i,1+i)$ (Ω-orbits in $\mathbb{C}$)

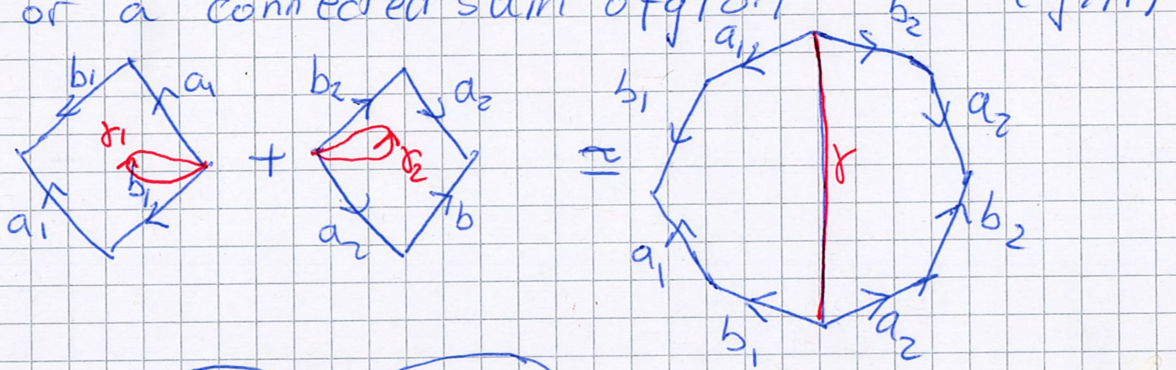
$\Omega(0,1,i,1+i) = \{m1 + ni \,/\, m, n \in \mathbb{Z}\}$ is a case of action of groups on an space (and of covering)

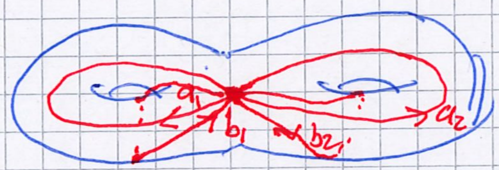
Consider X an space and G a group of homeomorphisms of X. Consider the space of G-orbits $Y = \{[x]_G\} = X/G$ the projection $p : X \to X/G$ $p(x) = [x]_G$ and $V \subseteq X/G$ open iff

~~For the Torus $p : X \to X/\Omega$~~

$p^{-1}(V)$ open in X. So p is continuous and open.
Now, for the torus T $[z]_\Lambda$ is discrete in $\mathbb{C}$. Let $0<d$ be the smallest distance between any two pts of $p^{-1}([z]_\Lambda)$ $(= z+\Lambda)$. Consider an open disc U of radius $d/2$ centered on any pt $\in p^{-1}([z]_\Lambda)$, then the map $p: U \to V$ bijective, continuous, open, i.e homeomorphism.

<u>Theorem</u> Any compact orientable surface is the sphere or a connected sum of g tori $(g \geq 1)$





We will see that when $g \geq 2$ the surfaces are quotients of the hyperbolic plane.

Euler characteristic $\chi(X) = 2-2g = \# v - \# e + \# f$
(v = vertex, e = edge, f = face)

- Recall also the fundamental group of a (arcconnected) space X, $\pi_1(X)$, is the group of homotopy classes of loops (based at one pt of the space, groups based at two different pts are isomorphic, we will omit the base pt)

Examples $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(S^2) = \{1d\}$, $\pi_1(\mathbb{C}) = \{1d\} = \pi_1(\mathbb{H}^2)$
$\pi_1(T) = \mathbb{Z} \times \mathbb{Z} = \{a, b \mid aba^{-1}b^{-1} = 1d\} = \Omega$
$\pi_1(T \oplus \overset{g}{\cdots} \oplus T) = \{a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_i [a_i, b_i] = 1d\}$

- Recall also that if $Z = X \times Y$ (still arcconnected) then $\pi_1(Z) = \pi_1(X) \times \pi_1(Y)$.

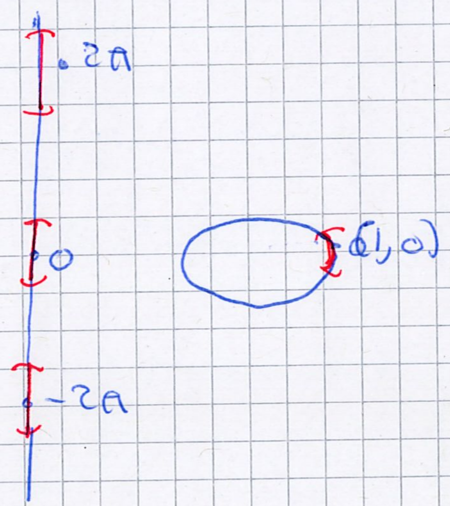
- In general Seifert-Van Kampen Theorem stablishes that if a space X is the union of two open, arcconnected subspaces U and V, then $\pi_1(X, x_0)$ is the free product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ with amalgamation $\pi_1(U \cap V, x_0)$

- The projection $\mathbb{C} \longrightarrow \mathbb{C}/\Omega = T$ is an example of covering. In general given a topological space X, a <u>covering</u> of X is a pair $(\tilde{X}, p)$ with $p : \tilde{X} \to X$ continuous s.t each point $x \in X$ has an arcconnected open neighbourhood U s.t each arc component of $p^{-1}(U)$ is mapped homeomorphically to U.

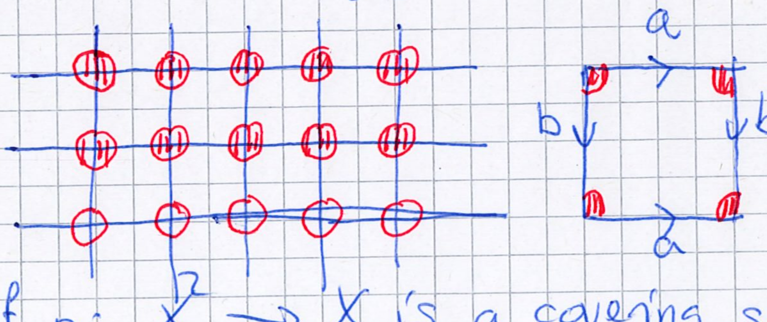
We have the examples

$p : \mathbb{R} \longrightarrow S^1$

$t \longmapsto (\cos t, \sin t)$



$p : \mathbb{C} \longrightarrow \mathbb{C}/\Omega$



If $p : \tilde{X} \to X$ is a covering space, then the sets $p^{-1}(x)$ for all $x \in X$, have the same cardinal number: <u>the degree of p</u>

* Branched Coverings (for surfaces). Let X and Y be (compact, orientable) surfaces, a branch covering $p: Y \to X$ is a covering except for a nowhere-dense set: the branch set (we will consider coverings of finite degree N, and the branch set to be finite)

For instance $p: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ $p(z) = z^2$ is a degree two covering of $\hat{\mathbb{C}}$ ramified on $z_0 = 0$ and $z_1 = \infty$.

<u>Example</u> Consider X a space (surface) and G a discrete subgroup of homeomorphisms acting properly discontinuously on X of X (or G acting freely on X, the case of $\mathbb{R}$ and $\mathbb{C}$), clearly

$$p: X \longrightarrow X/G \text{ is a (branched) covering.}$$

- For coverings spaces we have that $p: \tilde{X} \to X$ a covering of X, then $\pi_1(\tilde{X})$ isomorphic to a subgroup of $\pi_1(X)$ (considering are connected spaces)

Two coverings $p_1: \tilde{X}_1 \to X$, $p_2: \tilde{X}_2 \to X$ are isomorphic iff (given $x_1 \in \tilde{X}_1$, $x_2 \in \tilde{X}_2$, $p_1(x_1) = p_2(x_2) = x_0$) the subgroups $p_{1*}\pi_1(\tilde{X}_1, x_1)$ and $p_{2*}\pi_1(\tilde{X}_2, x_2)$ are conjugated in $\pi_1(X, x_0)$

As, S^2, $\mathbb{C} = \mathbb{R}^2$, $\mathbb{H}^2$ are simply-connected the only coverings of them are themselves. On the other hand if say $\pi: \mathbb{H}^2 \to X$ is a covering, given any other covering $p: Y \to X$ we have a covering $\tilde{p}: \mathbb{H}^2 \to Y$, so $\mathbb{H}^2$ is the <u>universal covering</u> of X

- The automorphism group $A(\tilde{X}, p)$ of a covering

$$p: \tilde{X} \to X$$

is naturally isomorphic to the group of automorphisms of $p^{-1}(x)$, $x \in X$. And for any point $x \in X$ and $\tilde{x} \in p^{-1}(x)$ $A(\tilde{X}, p)$ is isomorphic to $N(p_* \pi_1(\tilde{X}, \tilde{x})) / p_* \pi_1(\tilde{X}, \tilde{x})$, where $N(H)$ is the normalizer of the subgroup H. Notice that the <u>degree of p is the index</u>.

When $p_* \pi_1(\tilde{X}, \tilde{x})$ is normal in $\pi_1(X, x)$ the covering is said to be <u>regular</u> (or <u>Galois</u>). If $p : \tilde{X} \to X$ is a regular covering $A(\tilde{X}, p)$ isomorphic to $\pi_1(X, x) / p_* \pi_1(\tilde{X}, \tilde{x})$

For universal coverings $A(Y, \pi)$ ($\pi : Y \to X$ universal cov.) is $\pi_1(X)$.

<u>Examples</u>: We have seen that $\pi : \mathbb{C} \to T = \mathbb{C}/\Omega$

$A(\mathbb{C}, \pi) = \pi_1(T) = \Omega$

For $g \geq 2$ $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma_g$, $\Gamma_g = \langle a_1, \dots a_g, b_1, \dots b_g / \prod [a_i, b_i] = 1 \rangle$

$\Gamma_g = A(\mathbb{H}^2, \pi) = \pi_1(\mathbb{H}^2/\Gamma_g)$.

* Finally a couple of words on <u>topological groups</u>

A group G is a topological group if the maps $m : G \times G \to G$, $m(g, h) = gh$ and $i : G \to G$ $i(g) = g^{-1}$ are continuous. Notice that the map $m_g(x) = xg$ is homeomorphism

<u>Examples</u>: i) $(\mathbb{C}, +)$ is a topological group

ii) $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is topological

iii) $PSL(2, \mathbb{C})$ is a topological group, quotient of $GL(2, \mathbb{C})$

In general if G is a topological gr and H normal in G the (quotient) space G/H is a topological group.

iv) $\mathbb{C}$ top (Abelian) and Ω normal in $\mathbb{C}$, $T = \mathbb{C}/\Omega$ is a topological group

A subgroup Γ of a top. group G is discrete if there is U neighborhood of $\{id\}$ s.t $U \cap \Gamma = \{id\}$